Project 3

Due Saturday, December 7, 2013

Consider the following PDE:

$$\frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

1. (1 pt.) Consider solutions in the form of perturbations propagating with velocity v,

$$u(x,t) = f(x - vt). \tag{2}$$

Obtain an ODE for $f(z)$. Check that this ODE has solutions of the form

$$f(z) = \frac{C_1}{\cosh[C_2(z - z_0)]}. \tag{3}$$

Find C_1 and C_2 as a function of v (z_0, on the other hand, is an arbitrary constant). Note that C_1 can be both positive and negative! As you can see, solutions of the form (2)–(3) are localized perturbations, propagating at a constant speed without changing their shape, despite dispersion and nonlinearity of the equation. Such perturbations are called *solitons*, if they also have some additional properties, some of which you are going to study in this project.

2. (2 pts.) Write a program solving Eq. (1) numerically on the interval $[0, L]$, where L is an arbitrary length, with periodic boundary conditions. ~~PHY5340 students need to implement Dirichlet, Neumann, and periodic boundary conditions, both homogeneous (zero) and inhomogeneous (nonzero) in the first two cases. PHY4140 students can choose one of these types and homogeneous BCs will suffice. They will get up to 1 bonus point for implementing other types of BCs.~~ For your program, choose an appropriate **explicit** finite difference algorithm and explain your choice in the report.

3. (3 pts.) Estimate the stability of your algorithm analytically and study it numerically. Specifically, given the mesh step along the x axis, h, what is the maximum time step $\tau_{\max}$ for which the algorithm remains stable? For your analytical calculations, a crude "order-of-magnitude" estimate will suffice. For your numerical calculations, choose different values of h and different initial conditions [e.g., of type (3)]. Vary both h and the initial conditions

in reasonable limits (e.g., $|C_1| \sim 1$ and h small enough (but not too small) that the discretization of these initial conditions is accurate enough. Do you find that your numerical stability limit depends on your initial conditions? Is the h dependence observed numerically the same as predicted analytically? Do the numerical and analytical estimates agree, at least by order of magnitude? In your numerical calculations of $\tau_{\max}$, you will need to calculate it with the accuracy of $\sim 10\%$. For the rest of the project, choose the time step somewhat below $\tau_{\max}$ making it a function of h in agreement with your results. If you found that your $\tau_{\max}$ depends on the initial conditions, choose the maximum value of all $\tau_{\max}$ you have found for this purpose. Basically, the goal is to set the time step τ so that your calculation is guaranteed to be stable without making τ too small.

4. **(Optional.)** (10 pts.) Write a program that solves Eq. (1) using an **implicit** algorithm. Does the implicit algorithm allow you to use a larger time step? How much larger before either stability or accuracy start to suffer significantly? Given that the computational cost per time step may be different, does an implicit algorithm provide computational time savings? Considering also the extra time (if any) it took you to develop and debug implicit code, did it make sense from the practical point of view? For the rest of the project, you can use either your explicit code or your implicit code (but you may find it useful to do a few calculations with both to compare the results when answering this question).

5. (3 pts.) Using your program, verify Eqs. (2)–(3). Vary the height C_1 in reasonable limits (say, between 0.1 and 5). Choose the values of L and z_0 and the boundary conditions so that the effect of the boundaries is negligible. **PHY5340 students** will need to study how the accuracy depends on h for at least a few representative values of C_1. Come up with some **quantitative** estimates of the accuracy. **PHY4140 students** only need to choose one reasonable h (perhaps varying it as a function of C_1 as appropriate) and make sure that their results are reasonably accurate; they will get up to 2 bonus points for studying the h dependence. Do a few plots as appropriate. Completing this question should guide you in choosing appropriate parameters for the rest of this project.

6. (5 pts.) Study collisions of two solitons. Start with an initial condition corresponding to two solitons far enough from each other that they do not interact significantly initially ~~and also far enough from the walls that they do not influence the results significantly~~. Make sure you choose the parameter C_1 for both solitons in such a way that they move at different speeds and

collide ~~before crossing the boundaries~~. Choose a few representative values of C_1, make sure that you have at least one case where both C_1 are of the same sign and one where they have opposite signs. Plot a few snapshots of your collisions (you can combine different times in the same plot) and also plot the positions of all the maxima as a function of the time (keep in mind the number of maxima can change as the solitons collide). Describe your results qualitatively.

7. **(Optional for PHY4140 students.)** (4 pts.) What happens if you have an arbitrary initial condition, e.g., a peak that does not obey Eq. (3), or perhaps obeys that equation, but with a wrong relation between C_1 and C_2? Study different initial conditions and plot the evolution in a few interesting and representative cases. Can you give a **general** qualitative description of your results?

~~8. (Optional.) (2 pts.) Study collisions of a single soliton with boundaries for different initial conditions.~~

~~9~~8. **Optional.** (up to 5 pts.) There is also a soliton solution of Eq. (1) with a "raised background" (or a different vacuum state, as particle physicists would probably say):

$$f(z) = a + \frac{C_1}{\cosh[C_2(z - z_0)] + C_3}. \tag{4}$$

Find these solutions analytically and study them computationally repeating some of the same studies as above.

Bonus problems

If you want some additional bonus points, talk to me after you have substantially completed at least the mandatory parts of the above problem and I can suggest some bonus problems.