Project 2

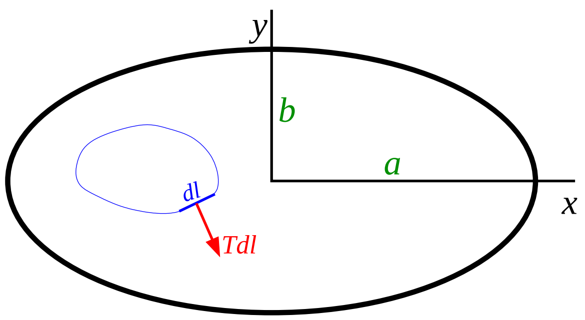
Due Saturday, November 23, 2013

Please choose one of the following two problems.

Problem 1

Consider a drum consisting of an elliptic frame with one semiaxis a and the other semiaxis b (see the figure) with a membrane stretched across it. You will need to find one or a few modes of vibrations of the membrane, as discussed below.

1. (2 pts.)Assume that the density (mass per unit area) ρ and the tension of the membrane are the same everywhere. The latter means that for any piece of the membrane, there is a force on the contour of the piece, with the force on a small element of the contour of length dl being Tdl, where $T = \text{const}$, and the direction of the force orthogonal to the direction of this element and parallel to the membrane (see the figure). Also, assume that the vibrations of the membrane are small (i.e., neglect all nonlinear effects). Derive the equation of motion of the membrane which should be a hyperbolic PDE. By separating the time variable from the space variables, reduce it to a 2D elliptic eigenvalue problem for the vibrational modes of the membrane. What other physics problems lead to the same equation? Assume $T = 1$ and $\rho = 1$ throughout the rest of this problem.



Please note that the resulting equation can be reduced to two ODEs by separation of variables in elliptic coordinates. **Hovewer**, for this problem you are **not allowed** to do that, you have to solve the 2D eigenvalue problem, although otherwise the method is up to you.

For the questions below, there are no specific requirements for the accuracy of your results (as long as the errors are not so high that the results are useless). But please do enough estimates of your error to give a good idea of your accuracy.

2. (8 pts.) Write a program that for given a and b finds the vibrational mode with the lowest frequency and the corresponding frequency. Find the shape of the mode and the frequency for $a = 1$, 2, 3, 5, and 10 ($b = 1$ in all cases). Do some plots to illustrate the modes. At the very least, plot the $x = 0$ and $y = 0$ cross sections of the modes, but also do other cross sections and/or 3D plots if you find them illustrative. Describe your results qualitatively.

3. **(Optional for PHY4140 students.)** (3 pts.) Calculating the frequency of the lowest mode for as many a as you deem necessary, plot the frequency as a function of a for $0.1 \le a \le 10$. Does the frequency seem to approach a constant value in the large-a limit? Why? What is the relation between the frequencies at a and $1/a$? Why?

4. **(Optional for PHY4140 students.)** (4 pts.)Because of the symmetry of the problem, all modes can be classified as either even or odd in both x and y directions (i.e., even-even, even-odd, odd-even, and odd-odd). (Of course, in case of degeneracy you may get a linear combination of modes of several classes.) For each of these classes, find the mode with the lowest frequency for the same values of a as in question 2. Do the necessary plots. Describe your results for both the shapes of the modes and the corresponding frequencies qualitatively.

5. **(Optional.)** (2 pts.) Plot the frequencies of the modes from question 4 as a function of a for a between 0.1 and 10. Describe the results qualitatively.

6. **(Optional.)** (5 pts.) For each of the values of a from question 2, find the 5 lowest modes (you will get partial credit for fewer than 5). Do the necessary plots. If you have done question 4, which of the modes coincide with those found in question 4 and which do not?

7. **(Optional.)** (7 pts.) For $0.1 \le a \le 10$, find the frequencies of the 10 lowest modes (you will get partial credit for fewer than 10) and plot **in a single plot**. If you have done question 5, which of the modes coincide with those found in question 5 in which intervals of a?

8. **(Optional for PHY4140 students.)** (1 pt.) Solve the equation analytically for $a = b = 1$. Your solution should be in the form of Bessel

functions. Compare the analytical result to all of your relevant numerical ones. Use any software you wish to calculate Bessel functions.

Problem 2

Consider a capacitor consisting of two infinitely thin identical conducting parallel plates in vacuum separated by distance d.

- **For PHY4140 students**, the plates are rectangular with an arbitrary half-width a that can be both larger and smaller than d and length L that is assumed much larger than both a and d throughout this problem.

- **For PHY5340 students**, the plates are discs of radius a that can be both larger and smaller than d.

Assume that the plates have equal and opposite charges $\pm q$. As always, the capacitance is calculated as the ratio of q and the potential difference between the plates U.

1. (1 pt.) Calculate the capacitance analytically in the limits of $d \ll a$ and $d \gg a$ (PHY4140 students, still assume $d \ll L$ in the latter case). For $d \gg a$ you are only required to find the a and d dependence, getting the right numerical prefactor is not necessary. You can use either CGS or SI units, but specify which you use.

2. (not marked separately) Write a program finding both the capacitance (for PHY4140 students, the capacitance per unit length L) and the charge distribution on the plates for given a and d. Consider both the finite difference and the boundary element methods (both described in class), but feel free to use any other method for solving PDEs. All your results for the capacitance should be accurate to within 1%.

3. (10 pts.) Find the capacitance and the charge distribution for $d = 0.1$ (optional, for 1 bonus point), $d = 0.2$, $d = 0.5$, $d = 1$, $d = 2$, $d = 5$, and $d = 10$ (the latter optional, for 1 bonus point). In all cases, $a = 1$. Estimate your error for both the capacitance and the charge distribution. Plot the charge distributions. Describe the results qualitatively and compare to both analytical results (for $d \ll a$ and $d \gg a$).

4. **(Optional for PHY4140 students.)** (3 pts.) Getting enough intermediate values of the capacitance, plot the dependence of the capacitance on d (keeping $a = 1$), for $0.2 < d < 5$. Extend to $d = 0.1$ and to $d = 10$ for up to 2 bonus points (depending on how many intermediate values you get).

5. **(Optional for PHY4140 students.)** (5 pts.) Plot the electric field lines for the same values of d as in question 3. Describe the result qualitatively.

6. **(Optional.)** (up to 10 pts.) Repeat at least some of your results with the other method (i.e., finite differences, if you used the boundary element method, or any of the two, if you used something else). You get at least 5 points for repeating any of your results and up to 10 points for repeating all of them at the same level of accuracy.