# Hyperbolic PDEs

Wave equation $\frac{\partial^2 u(\vec{r},t)}{\partial t^2} = c^2 \nabla^2 u(\vec{r},t)$

Need two initial conditions: $u(\vec{r},0) = f(\vec{r})$ and $\left.\frac{\partial u(\vec{r},t)}{\partial t}\right|_{t=0} = g(\vec{r})$

As before, boundary conditions, e.g.

$u(\vec{r},t)|_{\Gamma} = p(\vec{r},t)$ (Dirichlet) or $\left.\frac{\partial u(\vec{r},t)}{\partial n}\right|_{\Gamma} = p(\vec{r},t)$ (Neumann)

For mechanical waves, displacement is specified. For EM waves, e.g., a metallic surface

A force or pressure is specified, e.g., an approximation for the end of an open pipe

1D: $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$ $u(x,0) = f(x)$ and $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x)$

In free space (no boundaries), assuming $u(\pm\infty,t)=0$,

$$u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}[s(x+ct)-s(x-ct)], \quad s(x) = \int g(x')\,dx'$$

D'Alembert formula

$u(x,t) = F(x-ct) + G(x+ct)$ 2 travelling waves.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Fourier analysis. $u(x,t) = \exp(i(kx - \omega t))$

$$-\omega^2 \exp(i(kx-\omega t)) = -c^2 k^2 \exp(i(kx-\omega t)) \;\Rightarrow\; \omega = \pm ck$$

Two important features:

1. $\omega$ is real. Or, if we define $\lambda$, as in the previous lecture, i.e.,

$u(x,t) = \exp(ikx - \lambda t)$, then $\lambda = i\omega$ is purely imaginary. Modes do not

grow nor decay. Lack of dissipation.

2. $|\omega / k| = c = \text{const.}$ All modes propagate with the same speed.

Lack of dispersion.

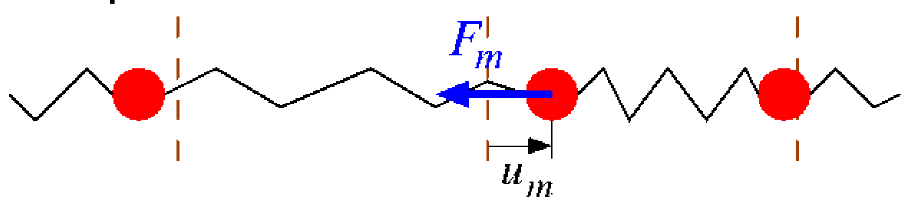
Most numerical schemes applied to the wave equation will have either dissipation (numerical diffusion, numerical viscosity), or dispersion, or both.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Use method of lines. Second-order discretization for the 2nd derivative.

$$\frac{d^2 u_m}{dt^2} = c^2 \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2}$$

These are actually Newton's equations of motion for a chain of beads connected by springs:




$$\mu F_m = -k(u_m - u_{m-1}) - k(u_m - u_{m+1}) = k(u_{m+1} - 2u_m + u_{m-1}) \qquad \frac{c^2}{h^2} = \frac{k}{\mu}$$

$$u_m = \exp(i(khm - \omega t))$$

$$-\omega^2 e^{i(khm-\omega t)} = \frac{c^2}{h^2}\left(e^{i(kh(m+1)-\omega t)} - 2e^{i(khm-\omega t)} + e^{i(kh(m-1)-\omega t)}\right)$$

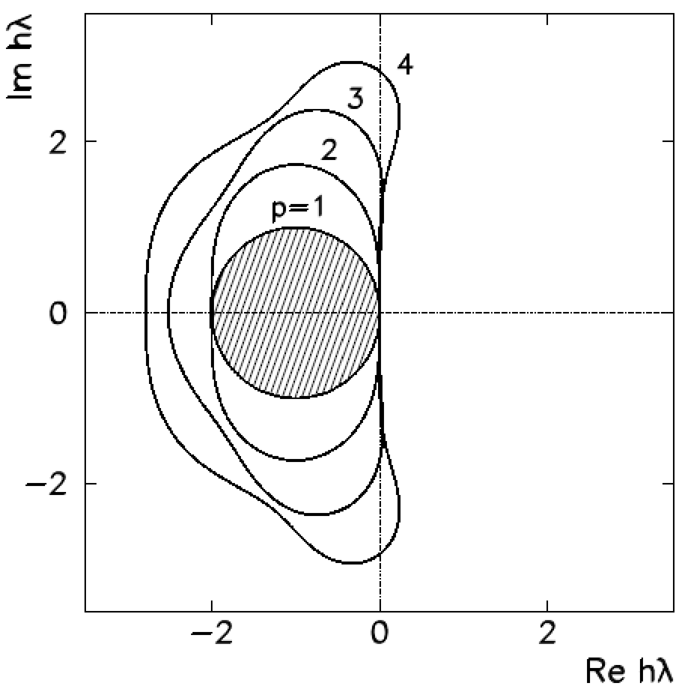
$$\omega^2 = \frac{c^2}{h^2}(2 - 2\cos(kh)) = \frac{4c^2}{h^2}\sin^2(kh/2) \;\Rightarrow\; \omega = \pm\frac{2c}{h}\sin(kh/2)$$

No damping ($\omega$ is real), but there is dispersion ( $\omega/k = c\left(1 - \frac{(kh)^2}{24} + O((kh)^4)\right)$ )

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\frac{d^2 u_m}{dt^2} = c^2 \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2}$$

$$u_m = \exp(i(khm - \omega t)) = \exp(ikhm - \lambda t)$$




Forward Euler is unstable, so unlike for the diffusion equation, FTCS is not a viable method.

RK4 is conditionally stable, but introduces a bit of damping. 4th order in time is overkill.

Some implicit methods are unconditionally stable: backward Euler, trapezoidal (Crank-Nicolson). The latter does not introduce any damping.

Verlet algorithm:

$$\frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{\tau^2} = c^2 \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}$$

$$\frac{u_m^{n+1}-2u_m^n+u_m^{n-1}}{\tau^2}=c^2\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}$$

$$u_m^{n+1}=2u_m^n-u_m^{n-1}+\frac{c^2\tau^2}{h^2}\left(u_{m+1}^n-2u_m^n+u_{m-1}^n\right)$$

Centred-time, centred-space scheme.

$$u_m=\exp\left(i\left(khm-\omega\tau n\right)\right)$$

$$e^{-i\omega\tau}=2-e^{i\omega\tau}+\frac{c^2\tau^2}{h^2}\left(e^{ikh}-2+e^{ikh}\right) \;\Rightarrow\; 2\sin^2(\omega\tau/2)=\frac{2c^2\tau^2}{h^2}\sin^2(kh/2)$$

$$\sin(\omega\tau/2)=\pm\frac{c\tau}{h}\sin(kh/2)$$

$\alpha=\frac{c\tau}{h}$ – Courant number

For $\alpha \le 1$, $\omega$ is real – stable, no dissipation. Moreover, for $\alpha = 1$ there is no dispersion either:

$$\sin(\omega\tau/2)=\pm\sin(kh/2) \;\Rightarrow\; \omega\tau=\pm kh \;\Rightarrow\; \omega=\pm\frac{kh}{\tau}=\pm ck \quad (k<\pi/h)$$

$$u_m^{n+1}=u_{m+1}^n+u_{m-1}^n-u_m^{n-1}$$

$$u_{m+1}^n=u_m^{n-1} \;\Rightarrow\; u_m^{n+1}=u_{m-1}^n$$

$$u_{m-1}^n=u_m^{n-1} \;\Rightarrow\; u_m^{n+1}=u_{m+1}^n$$

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$$\sin(\omega\tau/2) = \pm\frac{c\tau}{h}\sin(kh/2)$$

$$\alpha = \frac{c\tau}{h} > 1,\ k = \frac{\pi}{h}:\qquad e^{i\omega\tau/2} - e^{-i\omega\tau/2} = \pm 2i\alpha \qquad G \equiv e^{-i\omega\tau/2}$$

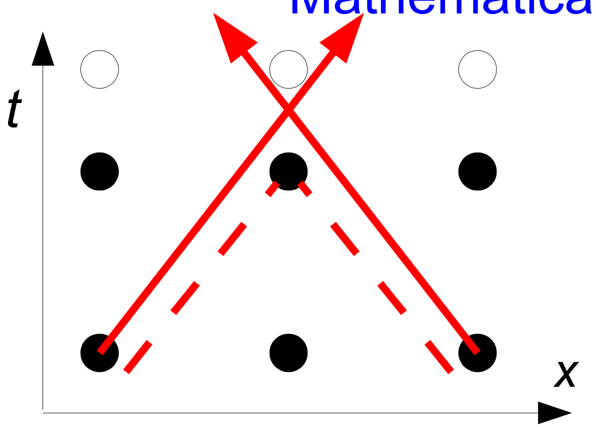
$$G - 1/G \pm 2i\alpha = 0 \ \Rightarrow\ G = i\left(\pm\alpha \pm\sqrt{\alpha^2 - 1}\right)$$

Growth factor $e^{-i\omega\tau} = G^2 = \left(\alpha \pm \sqrt{\alpha^2-1}\right)^2$ One of the two roots is > 1.

Unstable. Most accurate on the verge of instability.

$\frac{c\tau}{h} \le 1$ is a particular case of the Courant-Friedrichs-Lewy condition:

Mathematical domain of dependence is contained in numerical one.




The necessary value can be obtained by interpolation of the provided data.

$$u_m^{n+1} = 2u_m^n - u_m^{n-1} + \frac{c^2\tau^2}{h^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)$$

Need to know the values at 2 previous steps.

$$u(x,0) = f(x) \text{ and } \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x)$$

Initialization: $u_m^0 = f_m;\ u_m^1 = f_m + \tau g_m + \frac{c^2\tau^2}{2h^2}\left(f_{m+1}^n - 2f_m^n + f_{m-1}^n\right)$

Or use Velocity Verlet

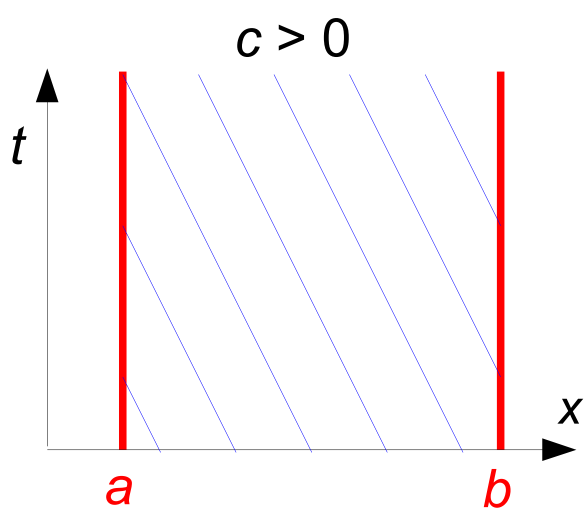
## An even simpler, 1st order wave equation:

$$\frac{\partial u(x,t)}{\partial t} = c\,\frac{\partial u(x,t)}{\partial x}$$

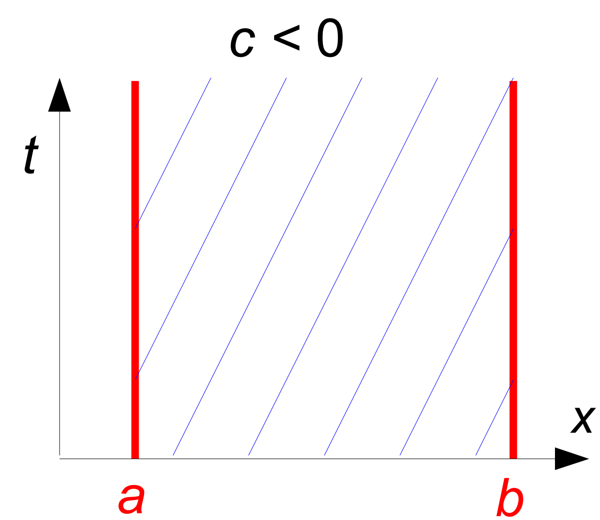
Also known as advection equation.

Need one initial condition: $u(x,0) = f(x)$

In free space, the solution is $u(x,t) = f(x+ct)$. Waves propagate in one direction. Lines *x*+*ct* = const are called characteristics.




Boundary condition:
$u(b,t) = p(t)$




Boundary condition:
$u(a,t) = p(t)$

$$\frac{\partial u(x,t)}{\partial t}=c\frac{\partial u(x,t)}{\partial x}$$

Can be generalized to systems of equations:

$$\frac{\partial \vec{u}(x,t)}{\partial t}=\boldsymbol{A}\frac{\partial \vec{u}(x,t)}{\partial x}$$

Diagonalizing ***A*** decouples these equations.

In particular, the 2nd order wave equation can be reduced to two 1st order equations.

Define $r=c\frac{\partial u}{\partial x}; \quad s=\frac{\partial u}{\partial t}.$

$$\frac{\partial r}{\partial t}-c\frac{\partial s}{\partial x}=c\frac{\partial^2 u}{\partial t\,\partial x}-c\frac{\partial^2 u}{\partial x\,\partial t}=0 \qquad \frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial s}{\partial t}-c\frac{\partial r}{\partial x}=0$$

$\vec{w}=(r,s) \qquad \frac{\partial \vec{w}}{\partial t}=\boldsymbol{A}\frac{\partial \vec{w}}{\partial x} \qquad \boldsymbol{A}=\begin{pmatrix}0 & c\\ c & 0\end{pmatrix}$ Gets diagonalized by considering

$$\frac{\partial(r+s)}{\partial t}=c\frac{\partial(r+s)}{\partial x}, \quad \frac{\partial(r-s)}{\partial t}=-c\frac{\partial(r-s)}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = v \qquad\qquad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

So the considerations for the 1$^{st}$ order wave eq. are relevant for the 2$^{nd}$ order eq. as well.

$$\frac{\partial u(x,t)}{\partial t} = c\frac{\partial u(x,t)}{\partial x} \qquad\qquad u(x,t) = \exp(i(kx - \omega t))$$

$$-i\omega \exp(i(kx-\omega t)) = ick\exp(i(kx-\omega t)) \;\Rightarrow\; \omega = -ck$$

Semidiscrete. Tempting to use 2$^{nd}$ order centred difference, but let's keep in mind 2$^{nd}$ order isn't always better.

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=x_m} \approx \beta\frac{u_{m+1}(t) - u_m(t)}{h} + (1-\beta)\frac{u_m(t) - u_{m-1}(t)}{h}.$$

$\beta = 0$ – left difference; $\beta = 1$ – right difference; $\beta = 1/2$ – centred difference.

$$\frac{\partial u(x,t)}{\partial t} = c\frac{\partial u(x,t)}{\partial x} \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=x_m} \approx \beta\frac{u_{m+1}(t)-u_m(t)}{h} + (1-\beta)\frac{u_m(t)-u_{m-1}(t)}{h}.$$

$$\frac{d\,u_m(t)}{dt} = c\left[\beta\frac{u_{m+1}(t)-u_m(t)}{h} + (1-\beta)\frac{u_m(t)-u_{m-1}(t)}{h}\right]$$

$$= c\left[\frac{u_{m+1}(t)-u_{m-1}(t)}{2h} + (2\beta-1)\frac{u_{m+1}(t)-2u_m(t)+u_{m-1}(t)}{2h}\right]$$

$$u_m(t) = \exp(i(khm-\omega t)) \qquad 2\beta-1 \equiv \gamma$$

$$-i\omega = \frac{c}{h}[i\sin(kh)+\gamma(\cos(kh)-1)] \qquad \omega = \frac{c}{h}[-\sin(kh)-2i\gamma\sin^2(kh/2)]$$

$$u_m(t) = \exp(ikhm-\lambda t) \qquad \lambda = i\omega = \frac{c}{h}[-i\sin(kh)+2\gamma\sin^2(kh/2)]$$

Numerical dissipation – what for? Stabilize unstable schemes like Euler.

$$\frac{d\,u_m(t)}{dt}=c\left[\frac{u_{m+1}(t)-u_{m-1}(t)}{2h}+\gamma\frac{u_{m+1}(t)-2u_m(t)+u_{m-1}(t)}{2h}\right]$$

$$\lambda=i\,\omega=\frac{c}{h}[-i\sin(kh)+2\gamma\sin^2(kh/2)]$$

$$\frac{d\,u_m(t)}{dt}\to\frac{u^{n+1}-u^n}{\tau}\qquad u_m^n=\exp(ikhm-\lambda n\tau)\qquad \lambda\to\frac{1-e^{-\lambda\tau}}{\tau}$$

$$\lambda=-\frac{1}{\tau}\ln\left\{1-\frac{c\tau}{h}[-i\sin(kh)+2\gamma\sin^2(kh/2)]\right\}$$

Stability condition: $|1-\alpha[-i\sin(kh)+2\gamma\sin^2(kh/2)]|\le 1$ $\qquad \frac{c\tau}{h}=\alpha$

$$[1-2\alpha\gamma\sin^2(kh/2)]^2+\alpha^2\sin^2(kh)\le 1$$

$$1-4\alpha\gamma\sin^2(kh/2)+4\alpha^2\gamma^2\sin^4(kh/2)+4\alpha^2\sin^2(kh/2)-4\alpha^2\sin^4(kh/2)\le 1$$

$$\sin^2(kh/2)\equiv\eta\qquad 4\alpha(\alpha-\gamma)\eta+4\alpha^2(\gamma^2-1)\eta^2\le 0\qquad \eta\ge 0$$

$$\alpha(\alpha-\gamma)+\alpha^2(\gamma^2-1)\eta\le 0$$

$$u_m^{n+1} = u_m^n + \alpha\left[\frac{u_{m+1}^n - u_{m-1}^n}{2} + \gamma\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2}\right]$$

$$\alpha(\alpha-\gamma) + \alpha^2(\gamma^2-1)\eta \le 0 \qquad\qquad 0 \le \eta \le 1$$

1. $\gamma = 0.$ Centred difference. $\alpha^2(1-\eta) > 0$ for any $\alpha \ne 0$

Unconditionally unstable. So indeed, the most accurate approximation is a bad choice.

2. $\gamma = -1.$ Left difference. $\alpha(\alpha+1) \le 0 \;\Rightarrow\; -1 \le \alpha \le 0$

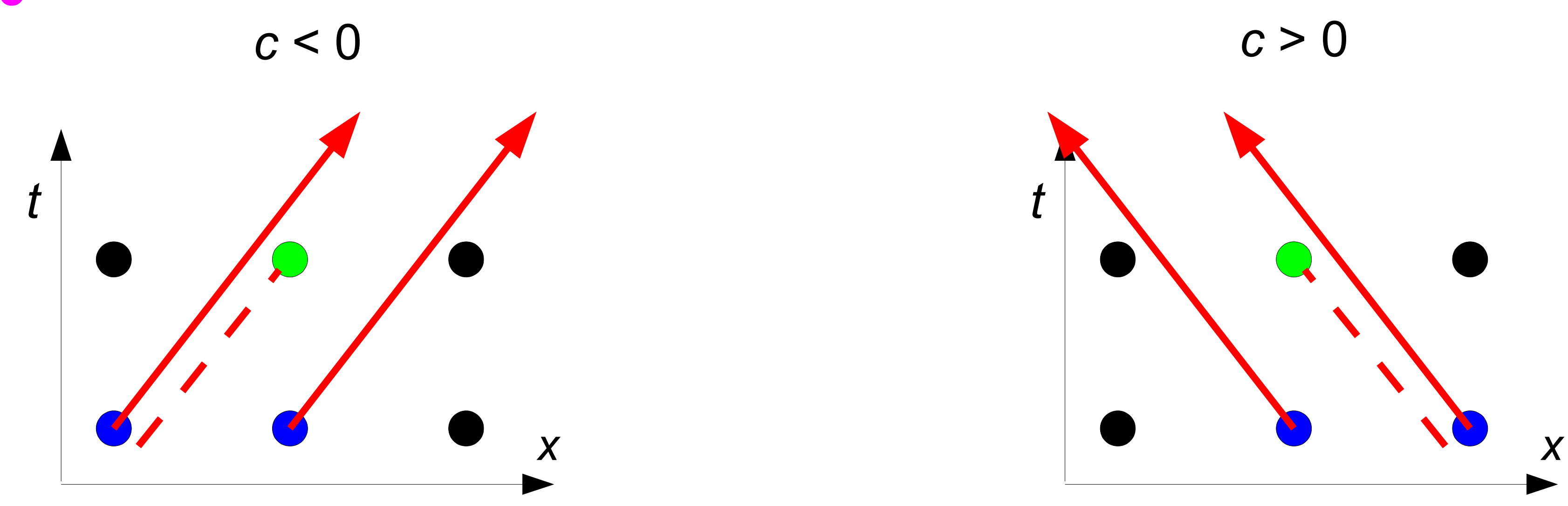
$\alpha \le 0 \;\Rightarrow$ stable when the wave travels to the right.

$|\alpha| \le 1$ – Courant-Friedrichs-Lewy condition

3. $\gamma = 1.$ Right difference. $\alpha(\alpha-1) \le 0 \;\Rightarrow\; 0 \le \alpha \le 1$

$\alpha \ge 0 \;\Rightarrow$ stable when the wave travels to the left.

$|\alpha| \le 1$ – Courant-Friedrichs-Lewy condition




Upwind scheme or forward-time forward-(backward-)space.

Can use higher-order space discretization, e.g.,

$$\frac{u^n_{m+1} - u^n_m}{h} \rightarrow \frac{-u^n_{m+2} + 4u^n_{m+1} - 3u^n_m}{2h}$$

$$u_m^{n+1} = u_m^n + \alpha \left[ \frac{u_{m+1}^n - u_{m-1}^n}{2} + \gamma \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2} \right]$$

$$\alpha(\alpha - \gamma) + \alpha^2(\gamma^2 - 1)\eta \le 0 \qquad\qquad 0 \le \eta \le 1$$

Is there a possibility to have a method stable for both $c > 0$ and $c < 0$?

4. $\gamma = 1/\alpha$.

(a) $|\alpha| < 1 \;:\; \gamma^2 - 1 > 0$ Maximum achieved for η = 1.

$\alpha(\alpha - 1/\alpha) + \alpha^2(1/\alpha^2 - 1) = 0.$ Right on the boundary.

(b) $|\alpha| > 1 \;:\; \gamma^2 - 1 < 0$ Maximum achieved for η = 0.

$\alpha(\alpha - 1/\alpha) = \alpha^2 - 1 > 0.$ Not satisfied. Thus, stable for $|\alpha| \le 1$ –

Courant-Friedrichs-Lewy condition

$$u_m^{n+1} = \frac{u_{m+1}^n + u_{m-1}^n}{2} + \alpha \frac{u_{m+1}^n - u_{m-1}^n}{2}$$

Lax-Friedrichs method. Strong diffusion, especially for smaller $\alpha$. Error is

$O(\tau) + O(h^2/\tau)$.

$$u_m^{n+1} = u_m^n + \alpha\left[\frac{u_{m+1}^n - u_{m-1}^n}{2} + \gamma\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2}\right]$$

$$\alpha(\alpha-\gamma) + \alpha^2(\gamma^2-1)\eta \le 0 \qquad\qquad 0 \le \eta \le 1$$

5. $\gamma = \alpha.$ $\qquad \alpha^2(\alpha^2-1)\eta \le 0 \;\Rightarrow\; |\alpha| \le 1.$

Note that, unlike the previous case, maximum is achieved for $\eta = 0$. More important low-wavelength modes do not decay.

$$\lambda = -\frac{1}{\tau}\ln\left\{1 - \alpha[-i\sin(kh) + 2\gamma\sin^2\frac{kh}{2}]\right\} = \frac{1}{\tau}\ln\left\{1 - i\alpha(kh) - 2\alpha\gamma\frac{(kh)^2}{4} + O((kh)^3)\right\}$$

$$= \frac{1}{\tau}\left\{-i\alpha(kh) - \alpha\gamma\frac{(kh)^2}{2} - \frac{1}{2}(-i\alpha(kh))^2 + O((kh)^3)\right\}$$

$$= \frac{1}{\tau}\left\{-i\alpha(kh) + (\alpha^2 - \alpha\gamma)\frac{(kh)^2}{2} + O((kh)^3)\right\}$$

Effectively, the method has quadratic accuracy in time and space.

Lax-Wendroff method.

$$u_m^{n+1} = u_m^n + \alpha\left[\frac{u_{m+1}^n - u_{m-1}^n}{2} + \gamma\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2}\right]$$

$$\alpha(\alpha-\gamma) + \alpha^2(\gamma^2-1)\eta \le 0$$

Common between these cases: $|\alpha| = 1, \quad \gamma = \alpha$

The left-hand side of the inequality is zero. No damping.

$\alpha = 1 : \quad u_m^{n+1} = u_{m+1}^n$ $\qquad\qquad$ $\alpha = -1 : \quad u_m^{n+1} = u_{m-1}^n$

Obviously, these are exact. But, of course, once there is any inhomogeneity or nonlinearity in the problem, this is no longer useful.

$$u_m^{n+1} = u_m^n + \alpha\left[\frac{u_{m+1}^n - u_{m-1}^n}{2} + \alpha\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2}\right]$$

An interesting way to interpret Lax-Wendroff:

$$u_m^{n+1} = \frac{\alpha(\alpha-1)}{2}u_{m-1}^n + (1-\alpha)(1+\alpha)u_m^n + \frac{\alpha(\alpha+1)}{2}u_{m+1}^n$$

$$= \frac{\alpha h(\alpha h - h)}{2h^2}u_{m-1}^n + \frac{(h-\alpha h)(h+\alpha h)}{h^2}u_m^n + \frac{\alpha h(\alpha h + h)}{2h^2}u_{m+1}^n$$

This is Lagrange interpolation for $u_{m+\alpha}^n$.

Can build an upwind scheme on the same principle. Assume $\alpha > 0$.

$$u_m^{n+1} = \frac{(h-\alpha h)(2h-\alpha h)}{2h^2}u_m^n + \frac{\alpha h(2h-\alpha h)}{h^2}u_{m+1}^n - \frac{\alpha h(h-\alpha h)}{2h^2}u_{m+2}^n$$

$$= \frac{(1-\alpha)(2-\alpha)}{2}u_m^n + \alpha(2-\alpha)u_{m+1}^n - \frac{\alpha(1-\alpha)}{2}u_{m+2}^n$$

Stable for $0 \le \alpha \le 2$. Beam-Warming method.

There is also a combination of Lax-Wendroff and Beam-Warming (Fromm).

# 18 Yet another interpretation of Lax-Wendroff:

$$u_m^{n+1} \approx u_m^n + \tau \left.\frac{\partial u}{\partial t}\right|_{x_m,t_n} + \frac{\tau^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x_m,t_n} = u_m^n + c\tau \left.\frac{\partial u}{\partial x}\right|_{x_m,t_n} + \frac{c^2\tau^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_m,t_n}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

$$u_m^{n+1} \approx u_m^n + \frac{c\tau}{2h}\left(u_{m+1}^n - u_{m-1}^n\right) + \frac{c^2\tau^2}{2h^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)$$

$$u_m^{n+1} = u_m^n + \alpha\left[\frac{u_{m+1}^n - u_{m-1}^n}{2} + \alpha\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2}\right] \qquad \alpha = \frac{c\tau}{h}$$

This also shows explicitly that the fact that Lax-Wendroff is 2nd order relies on the form of the equation and so its generalization to nonlinear equations is not straightforward.

**19** Can consider centred-time, centred-space method again. Leapfrog.

$$\frac{u_m^{n+1}-u_m^{n-1}}{\tau}=c\frac{u_{m+1}^{n}-u_{m-1}^{n}}{h}$$

$$u_m^{n+1}-u_m^{n-1}=\alpha\left(u_{m+1}^{n}-u_{m-1}^{n}\right)$$

$$u_m^n=\exp\left(i\left(khm-\omega n\tau\right)\right)$$

$$-2i\sin\left(\omega\tau\right)=2i\alpha\sin\left(kh\right)$$

$$\sin\left(\omega\tau\right)=-\alpha\sin\left(kh\right)$$

Stable for $|\alpha|\leq 1,$ no dissipation in this case. Mesh-drift instability for large gradients.