Singular value decomposition (SVD)

A general $m \times n$ matrix $\boldsymbol{A}$ (m rows, n columns) can always be represented as

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$$

$\boldsymbol{U}$ is $m \times m$ and orthogonal, $\boldsymbol{V}$ is $n \times n$ and orthogonal, $\boldsymbol{D}$ is $m \times n$ and diagonal:

$$\boldsymbol{D} = \begin{pmatrix} \sigma_1 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & 0 & \ldots & 0 \\ 0 & 0 & \sigma_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \sigma_n \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \text{ or } \boldsymbol{D} = \begin{pmatrix} \sigma_1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \sigma_3 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \sigma_m & 0 & \ldots & 0 \end{pmatrix}$$

σ_i are called the singular values, columns of U are left-singular vectors and those of V are right-singular vectors.

So $\boldsymbol{U}$ is the matrix of the eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$, $\boldsymbol{V}$ is the matrix of the eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$, and $\boldsymbol{D}$ contains the square roots of the corresponding eigenvalues of $\boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{A}^T\boldsymbol{A}$ (same, except extra eigenvalues are zero).

$$A = UDV^T$$

If $\boldsymbol{A}$ is square and symmetric, then $\boldsymbol{AA}^T = \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}^2$, the eigenvectors are the same as those of $\boldsymbol{A}$ and the eigenvalues are squares of those of $\boldsymbol{A}$. So up to a sign the singular values are the eigenvalues of $\boldsymbol{A}$ and the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$ are the eigenvectors. $\boldsymbol{U}$ and $\boldsymbol{V}$ are identical or can be chosen identical.

For general $\boldsymbol{A}$, $\quad A\vec{v}_i = \sigma_i \vec{u}_i \quad\quad A^T \vec{u}_i = \sigma_i \vec{v}_i$

For square and symmetric $\boldsymbol{A}$, in view of the above $A\vec{u}_i = \sigma_i \vec{u}_i$.

Returning again to the general case, for those i for which σ_i=0, $A\vec{v}_i = \vec{0}$.

So such $\vec{v}_i$ span the nullspace of $\boldsymbol{A}$: any $\vec{x}$ such that $A\vec{x} = \vec{0}$ is a linear combination of such $\vec{v}_i$.

Nullspace (or kernel) is spanned by the columns of $\boldsymbol{V}$ with corresponding σ=0.

The range of $\boldsymbol{A}$ (defined as the set of $\vec{b}$ such that $\exists \vec{x}: A\vec{x} = \vec{b}$) is spanned by the columns of $\boldsymbol{U}$ with corresponding $\sigma \neq 0$.

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$$

If ***A*** is square and invertible, then

$$\boldsymbol{A}^{-1} = (\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)^{-1} = \boldsymbol{V}\boldsymbol{D}^{-1}\boldsymbol{U}^T = \boldsymbol{V} \cdot \mathrm{diag}(\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_n^{-1}) \cdot \boldsymbol{U}^T$$

Invertible if and only if all $\sigma_i \neq 0$. Generalization: pseudoinverse

$$\boldsymbol{A}^+ = \boldsymbol{V}\boldsymbol{D}^+\boldsymbol{U}^T,$$

where $\boldsymbol{D}^+$ is an $n \times m$ diagonal matrix that has σ_i^{-1} on the diagonal, except infinities are replaced by zeros ($\infty \rightarrow 0$!).

$\boldsymbol{A}\vec{x} = \vec{b}$ has solutions, if and only if $\vec{b}$ is in the range of ***A***

If this is the case, then $\vec{x} = \vec{x}_0 = \boldsymbol{A}^+\vec{b}$ is a solution, not necessarily the only one

To this, any vector from the nullspace can be added, so the general solution is

$$\vec{x} = \boldsymbol{A}^+\vec{b} + \sum_{i:\sigma_i = 0} c_i \vec{v}_i$$

Solution $\vec{x} = \vec{x}_0 = \boldsymbol{A}^+\vec{b}$ is special, because it has the smallest norm of all.

$$\boldsymbol{A}\vec{x}=\vec{b}$$

If $\vec{b}$ is not in the range of ***A***, then there is no solution. However, $\vec{x}_0=\boldsymbol{A}^+\vec{b}$ minimizes the norm of the residual $\boldsymbol{A}\vec{x}-\vec{b}$. Consider $\vec{x}=\vec{x}_0+\vec{x}\,'$.

$$\boldsymbol{A}\vec{x}-\vec{b}=\boldsymbol{A}\boldsymbol{A}^+\vec{b}-\vec{b}+\boldsymbol{A}\vec{x}\,'=\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{D}^+\boldsymbol{U}^T\vec{b}-\vec{b}+\boldsymbol{A}\vec{x}\,'$$
$$=\boldsymbol{U}\boldsymbol{D}\boldsymbol{D}^+\boldsymbol{U}^T\vec{b}-\vec{b}+\boldsymbol{A}\vec{x}\,'$$

$$\boldsymbol{U}\boldsymbol{D}\boldsymbol{D}^+\boldsymbol{U}^T\vec{b}-\vec{b}=\sum_{i=1}^m\vec{u}_i s_i(\vec{u}_i\cdot\vec{b})-\sum_{i=1}^m\vec{u}_i(\vec{u}_i\cdot\vec{b})=\sum_{i=1}^m\vec{u}_i(s_i-1)(\vec{u}_i\cdot\vec{b}).$$

s_i = 1 or 0, with 0 only where σ_i = 0. So this is a linear combination of $\vec{u}_i$ with σ_i = 0. Conversely, $\boldsymbol{A}\vec{x}\,'$ belongs to the range of ***A*** and thus is a linear combination of $\vec{u}_i$ with $\sigma_i\neq 0$. So $\boldsymbol{A}\vec{x}_0-\vec{b}$ and $\boldsymbol{A}\vec{x}\,'$ are mutually orthogonal and the minimum norm is achieved when $\vec{x}\,'=\vec{0}$.

Often, it is found numerically that all singular values are nonzero, but some are very small, such that the ratio between the largest and the smallest by absolute value becomes comparable to machine precision. The corresponding entries in $\boldsymbol{D}^{+}$ will be very large. The uncertainty of such small singular values is large; it is impossible to be sure that they are not zero. The rank of the matrix is not very well defined.

In such cases, it may actually be better to introduce a cutoff and treat all singular values below the cutoff as zero.

$$\vec{x} = \boldsymbol{A}^{+}\vec{b} = \boldsymbol{V}\boldsymbol{D}^{+}\boldsymbol{U}^{T}\vec{b}$$

Subtracts some large but completely unreliable components of $\vec{x}$.

This may actually make both the error of the solution and the residual smaller.

This is a situation where using SVD may be preferable to direct methods of solving linear systems.

Other applications:

Linear least square fitting. Reduced to solving an overdetermined system.

Principal component analysis – directions of largest variance of data.

Approximation of a matrix by a lower rank matrix.

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$$

$$\boldsymbol{A}=\sum_{k=1}^{\min(m,n)}\sigma_k\vec{u}_k\otimes\vec{v}_k \qquad a_{ij}=\sum_{k=1}^{\min(m,n)}\sigma_k u_{k,i}v_{k,j}$$

Each term is a rank-1 matrix.

Matrix A can be represented as a collection of $\vec{u}_k,\ \vec{v}_k,\ \sigma_k$.

Neglect terms with small σ. Reduces storage requirements. (Lossy) data compression. Suitable for images.

512x512

Fig. 6.5. The original image (*left*) and those obtained using the first 20 (*center*) and 60 (*right*) singular values, respectively

A. Quarteroni, F. Saleri, P. Gervasio, Scientific Computing with MATLAB and Octave, Springer, Berlin, 2010.

Computing SVD

Eigenvalue decomposition for symmetric matrices: $\boldsymbol{Q}^T \boldsymbol{A}\boldsymbol{Q}$

First apply a series of orthogonal transformations to transform into a tridiagonal matrix. $\boldsymbol{A} \rightarrow \boldsymbol{Q}_1^T \boldsymbol{A}\boldsymbol{Q}_1 \rightarrow \boldsymbol{Q}_2^T \boldsymbol{Q}_1^T \boldsymbol{A}\boldsymbol{Q}_1 \boldsymbol{Q}_2 \rightarrow \ldots \rightarrow \boldsymbol{Q}_m^T \ldots \boldsymbol{Q}_1^T \boldsymbol{A}\boldsymbol{Q}_1 \ldots \boldsymbol{Q}_m$

Then diagonalize. For SVD (Golub-Kahan algorithm):

1. Bidiagonalization using Householder reflections.

$$\boldsymbol{U}^T \boldsymbol{A}\boldsymbol{V} = \begin{pmatrix} d_1 & f_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & d_2 & f_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & d_3 & f_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & d_{n-1} & f_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & d_n \end{pmatrix}$$

2. Diagonalization using Givens rotations.

Partial differential equations (PDEs)

An equation for an unknown function of at least 2 variables relating the function itself and its derivatives of various orders.

Function $u(x_{1,}..,x_n)$

$$F\left(x_{1,}..,x_n,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1 x_2},\ldots,\frac{\partial^2 u}{\partial x_1 x_n},\frac{\partial^2 u}{\partial x_2 x_3},\ldots\right)=0$$

Obviously, a huge variety. But most equations encountered in physics belong to a few much more narrow categories. A particularly interesting case is 2nd order equations linear with respect to the 2nd derivatives. In 2D,

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

$$A(x,y)u_{xx}+2B(x,y)u_{xy}+C(x,y)u_{yy}+f\left(x,y,u,u_x,u_y\right)=0$$

For simplicity, assume *A*, *B*, *C* = const. If we do an orthogonal variable transformation (e.g., rotation), the properties should not change.

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}u$$

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}q_{xx} & q_{xy}\\ q_{xy} & q_{yy}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial}{\partial y'}\frac{\partial y'}{\partial x}=\frac{\partial}{\partial x'}q_{xx}+\frac{\partial}{\partial y'}q_{xy}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial x'}\frac{\partial x'}{\partial y}+\frac{\partial}{\partial y'}\frac{\partial y'}{\partial y}=\frac{\partial}{\partial x'}q_{xy}+\frac{\partial}{\partial y'}q_{yy}$$

$$\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}=\begin{pmatrix}q_{xx} & q_{xy}\\ q_{xy} & q_{yy}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x'}\\ \frac{\partial}{\partial y'}\end{pmatrix}=\boldsymbol{Q}\begin{pmatrix}\frac{\partial}{\partial x'}\\ \frac{\partial}{\partial y'}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x'} & \frac{\partial}{\partial y'}\end{pmatrix}\boldsymbol{Q}^T\begin{pmatrix}A & B\\ B & C\end{pmatrix}\boldsymbol{Q}\begin{pmatrix}\frac{\partial}{\partial x'}\\ \frac{\partial}{\partial y'}\end{pmatrix}$$

The determinant of the matrix, $AC-B^2$, is invariant under such a transformation.
Can be diagonalized (B=0).

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+f\left(x,y,u,u_x,u_y\right)=0$$

$$AC-B^2>0: \quad au_{xx}+cu_{yy}+f\left(x,y,u,u_x,u_y\right)=0, \quad a,c>0$$

$$u_{xx}+u_{yy}+f\left(x,y,u,u_x,u_y\right)=0$$ Elliptic

$$AC-B^2<0: \quad au_{xx}-cu_{yy}+f\left(x,y,u,u_x,u_y\right)=0, \quad a,c>0$$

$$u_{xx}-u_{yy}+f\left(x,y,u,u_x,u_y\right)=0$$ Hyperbolic

$$AC-B^2=0: \quad au_{yy}+f\left(x,y,u,u_x,u_y\right)=0$$

$$-u_x+u_{yy}+f\left(x,y,u,u_y\right)=0$$ Parabolic

Names are based on the analogy with the equations for conic sections:

$$Ax^2+2Bxy+Cy^2+Dx+Ey+G=0$$

Can be generalized to higher dimensions:

Elliptic: $u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_n x_n}$ (all + signs)

Hyperbolic: $-u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_n x_n}$ (one - sign)

Parabolic: $-u_{x_1} + u_{x_2 x_2} + \ldots + u_{x_n x_n}$

Obviously, this does not exhaust all possibilities. E.g., ultrahyperbolic – has equal numbers of +'s and -'s. But the above 3 are the most important in physics.

What is special about each of these classes?

Parabolic and hyperbolic equations normally arise in time evolution problems and describe propagation of some perturbation

Hyperbolic: wave equation $u_{tt} = c^2 u_{xx}$

$$u(x, t=0) = f(x),\ u_t(x, t=0) = 0 \ \Rightarrow \ u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$$

Information propagates with speed *c*.

Parabolic: diffusion equation $u_t = D u_{xx}$ Note sign – otherwise backward

$$u(x, t=0) = \delta(x) \ \Rightarrow \ u(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

Information propagates instantaneously.

Generalization of the concept of hyperbolic equation to equations that are not even 2nd order or even systems of equations. E.g., $u_t = \pm c u_x$

$$u(x, t=0) = f(x) \ \Rightarrow \ u(x,t) = f(x \pm ct)$$

Appropriate numerical methods are time-marching methods.

Elliptic equations describe equilibrium or a steady state

$$u_t = D\left(u_{xx} + u_{yy}\right) \qquad u_t = 0 \;\Rightarrow\; u_{xx} + u_{yy} = 0 \text{ - Laplace equation}$$

$$u(x, y, t) = f(x, y)\, e^{-\lambda t} \;\Rightarrow\; D\left(f_{xx} + f_{yy}\right) = -\lambda f$$

Helmholtz equation – eigenvalue problem

Need to consider the whole solution at once, rather than in “slices”.

Can consider the auxiliary parabolic equation, start with an arbitrary initial condition and evolve in time until it relaxes towards equilibrium. Relaxation methods.

Since equations of different types arise in different situations, it is rare to have equations that are of different types in different regions, but it does happen. Example: Euler-Tricomi equation for transonic flow:

$$u_{xx} = x u_{yy}$$

Additional conditions needed to have a unique solution

Parabolic $u_t = D u_{xx}$

Initial condition: $u(x, t=0) = f(x)$

Boundary conditions: $u(x=x^{(1)}, t) = g_1(t),\ u(x=x^{(2)}, t) = g_2(t),$ or

$u'(x=x^{(1)}, t) = g_1(t),\ u'(x=x^{(2)}, t) = g_2(t),$ or a combination

Hyperbolic $u_{tt} = c^2 u_{xx}$

Initial conditions: $u(x, t=0) = f_1(x) \quad u'(x, t=0) = f_2(x)$

Same boundary conditions.

For more than 1 spatial dimension $[u_t = D(u_{xx} + u_{yy});\ u_{tt} = c^2(u_{xx} + u_{yy})]$,

$$u(x, y, t)\big|_S = g(t) \text{ or } \left.\frac{\partial u(x, y, t)}{\partial n}\right|_S = g(t)$$

Same for elliptic equations $[u_{xx} + u_{yy} = 0]$, except without *t*

Examples of physics PDEs

Wave equations: hyperbolic. String under tension

Consider small displacements, assume tension T = const, α = const.

$$T u_x(x+dx) - T u_x(x) = T u_{xx}\, dx = \alpha\, dx\, u_{tt}$$

$$u_{tt} = \frac{T}{\alpha} u_{xx}$$

α is linear mass density

Longitudinal deformations of an elastic rod:

$$\sigma = E\,\epsilon$$

$$\sigma(x) = F(x)/S \qquad \epsilon(x) = ([x+dx+u(x+dx)] - [x+u(x)] - dx)/dx = u_x(x)$$

$$F(x) = ES u_x(x)$$

$$\rho\, S\, dx\, u_{tt} = F(x+dx) - F(x) = ES[u_x(x+dx) - u_x(x)] = ES u_{xx}$$

$$u_{tt} = \frac{E}{\rho} u_{xx}$$

An elastic solid

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad \sigma_{ij} = C_{ijkl} \epsilon_{kl}, \ \epsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

For isotropic solids, $\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$ Lamé parameters

$$\mu \frac{\partial^2 u_i}{\partial x_j^2} + (\mu+\lambda) \frac{\partial^2 u_j}{\partial x_i x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\mu \nabla^2 \vec{u} + (\mu+\lambda) \nabla (\nabla \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \ \Rightarrow \ \text{eigenvalue problem}$$

Transverse: $u_{tt} = \frac{\mu}{\rho} u_{xx}$ Longitudinal: $u_{tt} = \frac{2\mu+\lambda}{\rho} u_{xx}$

$$2\mu + \lambda = \frac{1-\nu}{(1+\nu)(1-2\nu)} E$$

On the other hand, for beam bending

$$u_{tt} = -\frac{EI}{\alpha} u_{xxxx} \qquad I = \int\int z^2 \, dy \, dz$$

Like wave equation, but with dispersion

Maxwell's equations (Gaussian units)

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c}\left(4\pi \vec{J} + \frac{\partial \vec{E}}{\partial t}\right)$$

$$\nabla \times \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial\left(\nabla \times \vec{B}\right)}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c}\frac{\partial \vec{J}}{\partial t}$$

$$\nabla \times \nabla \times \vec{B} = \frac{1}{c}\frac{\partial\left(\nabla \times \vec{E}\right)}{\partial t} + \frac{4\pi}{c}\nabla \times \vec{J} = -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \frac{4\pi}{c}\nabla \times \vec{J}$$

Hyperbolic.

Statics: $\nabla \times \vec{E} = 0 \;\Rightarrow\; \vec{E} = -\nabla\phi \;\Rightarrow\; \nabla^2\phi = -4\pi\rho$

Poisson equation Elliptic

Many equations in physics follow from conservation laws.

General form:

Conserved quantity *M*, π is its density, $\vec{J}$ is the flux. $$\frac{\partial \pi}{\partial t} + \nabla \cdot \vec{J}[\pi] = 0$$

Mass conservation: $\pi = \rho$ $\vec{J} = \rho \vec{v}$ $$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v})$$

First-order equation, hyperbolic. If we take diffusion into account,

$$\vec{J} = \rho \vec{v} - D \nabla \rho \qquad \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v}) + D \nabla^2 \rho$$

Advection-diffusion equation, parabolic.

Momentum conservation (1D): $\pi = \rho v$ $J = \rho v^2$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial}{\partial x}(\rho v^2) = v \frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial t} + v \frac{\partial (\rho v)}{\partial x} + \rho v \frac{\partial v}{\partial x} = \rho \left[\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right] = 0$$

Burgers equation. Hyperbolic. Has shocks.

For conservation law equations it is very desirable to have methods that ensure the quantity that is supposed to be conserved is indeed conserved

Time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t)-U(\vec{r})\psi(\vec{r},t)=0$$

Looks parabolic, but because of *i*, has as solutions waves that don't decay, like hyperbolic equations. So, is it parabolic?

$$U=0;\ \psi=\exp[i(kx-\omega t)]$$

$$\hbar\omega-\frac{\hbar^2k^2}{2m}=0$$

Phase velocity $\frac{\omega}{k}=\frac{\hbar k}{2m}$

Can be arbitrarily high, which is a property of parabolic equations.

Time-independent Schrödinger equation

$$\psi(\vec{r},t)=\psi(\vec{r})\exp(-iEt/\hbar)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r})+U(\vec{r})\psi(\vec{r})=E\psi(\vec{r})$$

Hydrodynamics (Navier-Stokes equation)

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=-\nabla p+\eta\nabla^2\vec{v}+(\zeta+\eta/3)\nabla(\nabla\cdot\vec{v})$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$$

$$p=p(\rho)$$

The system as a whole does not belong to a particular category in general, but does in certain limits etc.

Sound propagation

$$\vec{v}=v(x,t)\vec{e}_x \qquad \rho=\rho_0+\delta\rho(x,t) \qquad p=p_0+\frac{K}{\rho_0}\delta\rho(x,t)$$

$$\rho_0\frac{\partial v}{\partial t}=-\frac{K}{\rho_0}\frac{\partial \rho}{\partial x}+(\zeta+4\eta/3)\frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial \rho}{\partial t}+\rho_0\frac{\partial v}{\partial x}=0$$

This is actually a vector conservation law.

$$\rho_0\frac{\partial^2 v}{\partial t^2}=-\frac{K}{\rho_0}\frac{\partial^2 \rho}{\partial x\partial t}+(\zeta+4\eta/3)\frac{\partial^3 v}{\partial x^2\partial t} \qquad \frac{\partial^2 \rho}{\partial x\partial t}+\rho_0\frac{\partial^2 v}{\partial x^2}=0$$

$$\rho_0\frac{\partial^2 v}{\partial t^2}=K\frac{\partial^2 v}{\partial x^2}+(\zeta+4\eta/3)\frac{\partial^3 v}{\partial x^2\partial t}$$

Hyperbolic.

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=-\nabla p+\eta\nabla^2\vec{v}+(\zeta+\eta/3)\nabla(\nabla\cdot\vec{v})$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \qquad\qquad p=p(\rho)$$

$$\vec{v}=v(y,t)\vec{e}_x \qquad \rho\frac{\partial v}{\partial t}=\eta\nabla^2\vec{v}$$ Parabolic. Momentum diffusion.

Potential flow: assume $\nabla\times\vec{v}=0$ and $\rho=\text{const.}$ $\qquad \vec{v}=\nabla\phi$

$$\nabla\cdot\vec{v}=0 \;\Rightarrow\; \nabla^2\phi=0$$ Elliptic.