Lab 3

Problem 1

Methods for solving the ODE:

Most people used either RK4 or Velocity Verlet. Both are OK in principle. Some of those using VV considered it very important to use a symplectic method – not really true in this case!

Order of accuracy is more important here. RK4 seems to have an advantage, but in most submissions it is destroyed by using a low-order "root-finding" procedure to find the period.

One person used the Euler method justifying it by saying that the error of finding the period itself is O(h) so it does not make sense to use a higher-order method. This is a valid argument in principle (more about that in a moment), but this O(h) error is, of course, not inevitable!

One other person did something I found strange: he basically repeated the previous lab doing the same integral for the period, but using the RK4 method for doing that integral!

$$\frac{dt}{dx} = 1/v = \frac{m}{2\sqrt{E(A) - U(x)}} \qquad E(A) = \alpha A^2 + \beta A^4$$

$$x = A \sin\theta$$

$$\frac{dt}{d\theta} = \frac{m}{2\sqrt{\alpha + \beta A^2 (1 + \sin^2\theta)}}$$

Doing integrals using an ODE method is, of course, possible, but generally “overkill”.

One minor point: using x=A, v=0 initial conditions instead of what was required.

Period finding. Most people found either the point where x(t)=0 or where it is a maximum (v(t)=0).

For those who used RK4, since it's a 4^{th} order method, it makes sense to find the crossing point with $O(h^4)$ accuracy as well. Otherwise, what's the point of using a high-order method?

Some used linear interpolation: $$t = t_i - \frac{x_i h}{x_{i+1} - x_i}$$

This is $O(h^2)$. While not matching RK4 accuracy, it's decent.

Many used some O(h) approach. E.g., just used the first time step after crossing; first step x decreases; average between the times a threshold is crossed. 10^6 steps per period needed – a lot! There will be a penalty for this. Some of these people did fewer steps, so did not quite achieve the required accuracy.

How to detect crossings to $O(h^4)$: interpolate with a cubic polynomial

Store 3 points at all times; if the 2nd and 3rd have opposite signs, store one more.

$$x_{0,} x_{1,} x_{2,} x_3$$

A straightforward approach, but not very good: interpolate x(t).

$$x(t) \approx x_0 \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} + x_1 \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}$$

$$+ x_2 \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} + x_3 \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)}$$

Will need to solve the cubic equation then.

Much better: interpolate t(x) instead of x(t).

$$t(x) \approx t_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + t_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ t_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + t_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Simply calculate this expression for $x = 0$!

100 steps per period is sufficient.

Root-finding methods: secant, bisection.

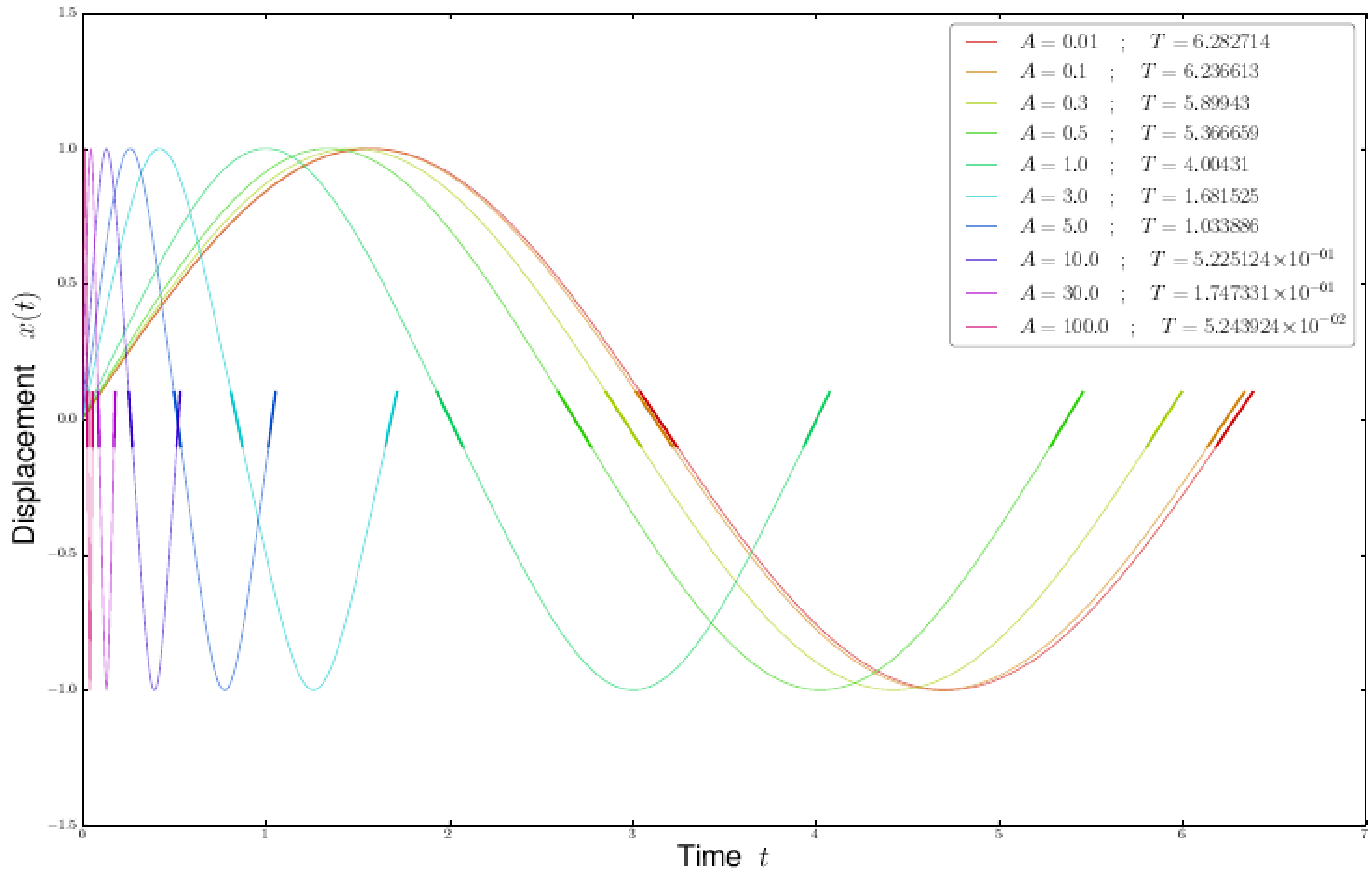
3 people did this, 2 of them had “deficiencies”, although the result is correct.

No need to integrate from the beginning, just do 1 step!

For bisection: only 1 calculation per iteration needed, not 3!

A very peculiar method: collect all data points within 0.1 of x=0 and do a linear fit. Not strictly banned, but ...

Works here (probably quadratic), but only because of the symmetry of the problem!



$A = 0.01$; $T = 6.282714$
$A = 0.1$; $T = 6.236613$
$A = 0.3$; $T = 5.89943$
$A = 0.5$; $T = 5.366659$
$A = 1.0$; $T = 4.00431$
$A = 3.0$; $T = 1.681525$
$A = 5.0$; $T = 1.033886$
$A = 10.0$; $T = 5.225124 \times 10^{-01}$
$A = 30.0$; $T = 1.747331 \times 10^{-01}$
$A = 100.0$; $T = 5.243924 \times 10^{-02}$
Displacement $x(t)$
Time t

Methods that are not satisfactory/did not work

1) Using FFT – wrong on several counts!

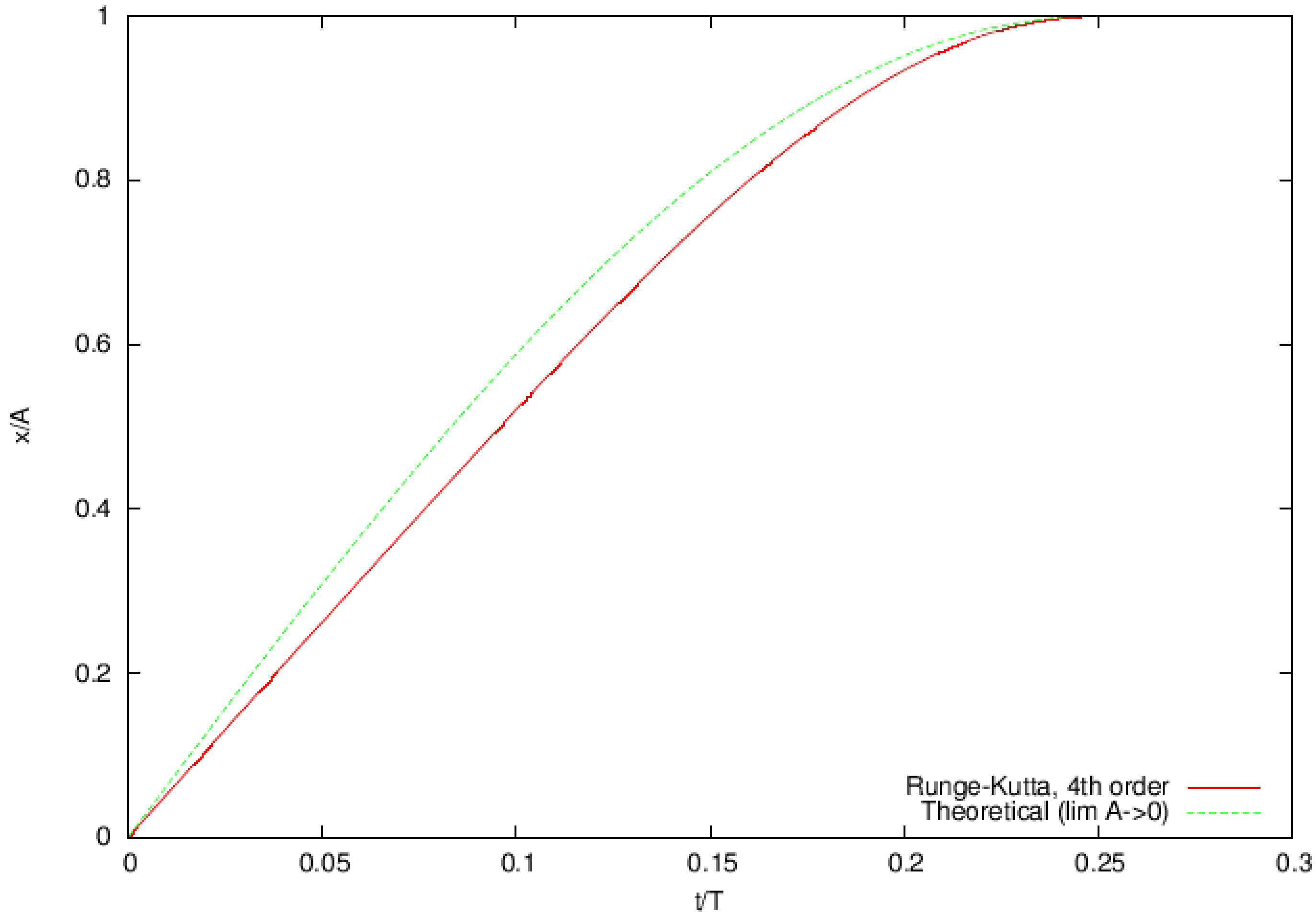
2) Finding the time x exceeds A for the first time. Does not necessarily happen, e.g., for RK4. This person used Euler, which is unstable, so the amplitude does grow, but still does not happen reliably. So the output looks like this:

```
1.570677900000000e+000 9.999999999999898e-003 1.000000000000058e-002
1.559153000000000e+000 9.999999999991972e-002 1.000000000000064e-001
1.371617410000000e+002 2.999999999998660e-001 3.000000000000112e-001
1.341665000000000e+000 4.999999999998147e-001 5.000000000000688e-001
5.005387000000000e+000 9.999999999989641e-001 1.000000000000171e+000
4.203810000000000e-001 2.999999999959050e+000 3.000000000000899e+000
2.584716000000000e-001 4.999999999998257e+000 5.000000000000097e+000
1.306281000000000e-001 9.999999999991323e+000 1.000000000000067e+001
4.368330000000000e-002 2.999999999983049e+001 3.000000000001687e+001
```

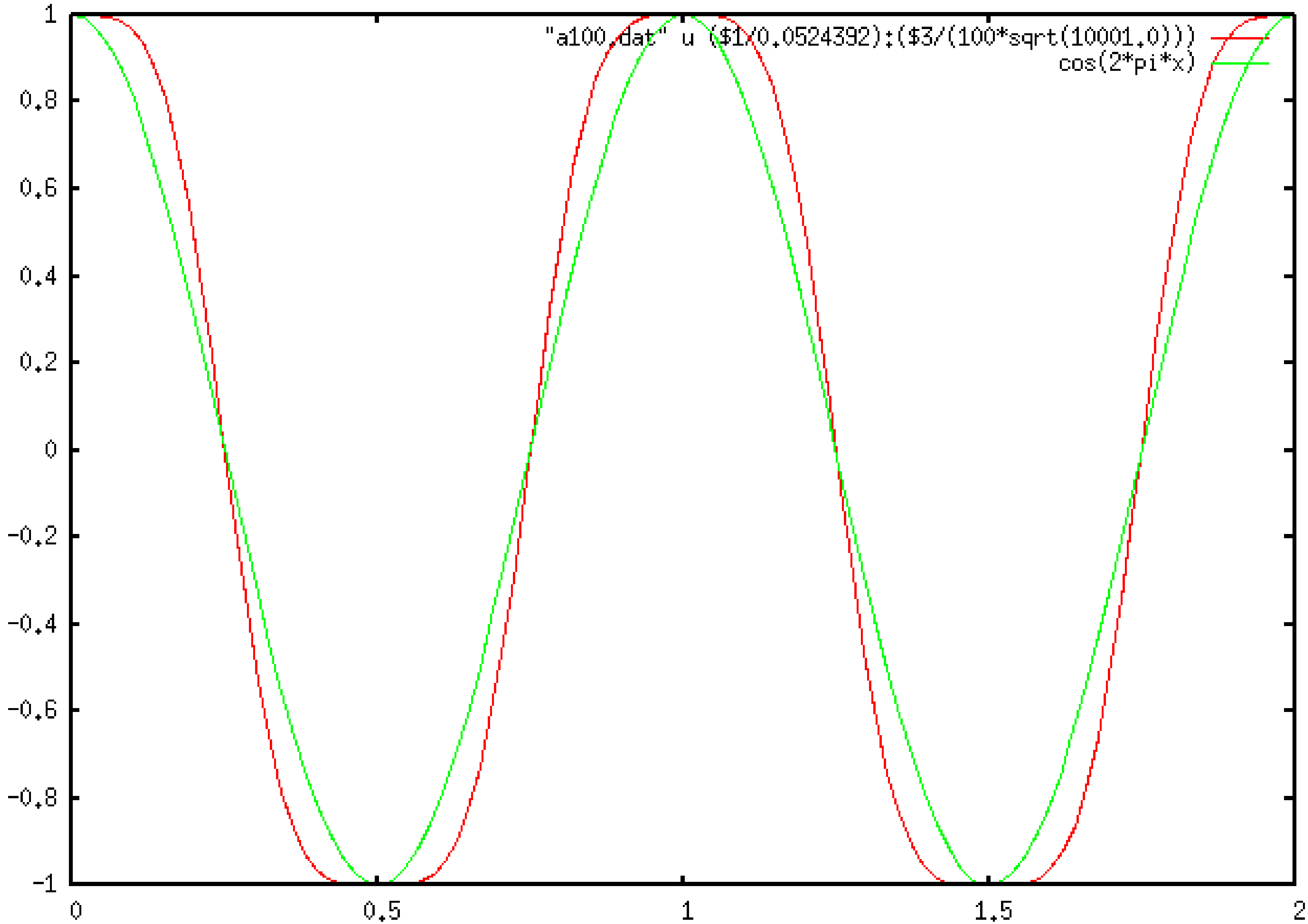
3) Find the time when x first exceeds a threshold slightly below A.

Why is this not a good idea?

Compare x(t) for A→0 and A=100.



More pronounced for the velocity!



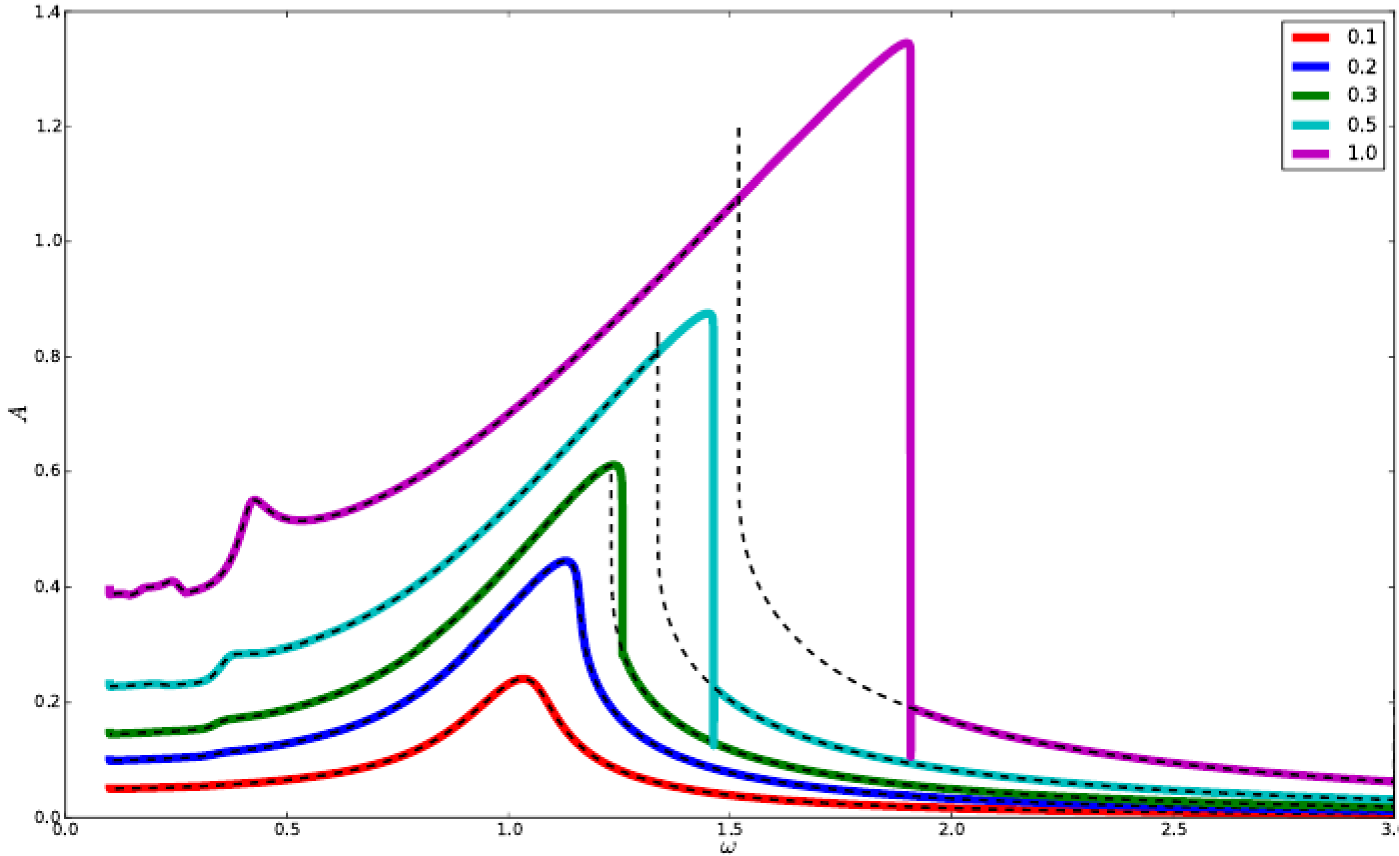
The person who did this will get bonus points!

Problem 2

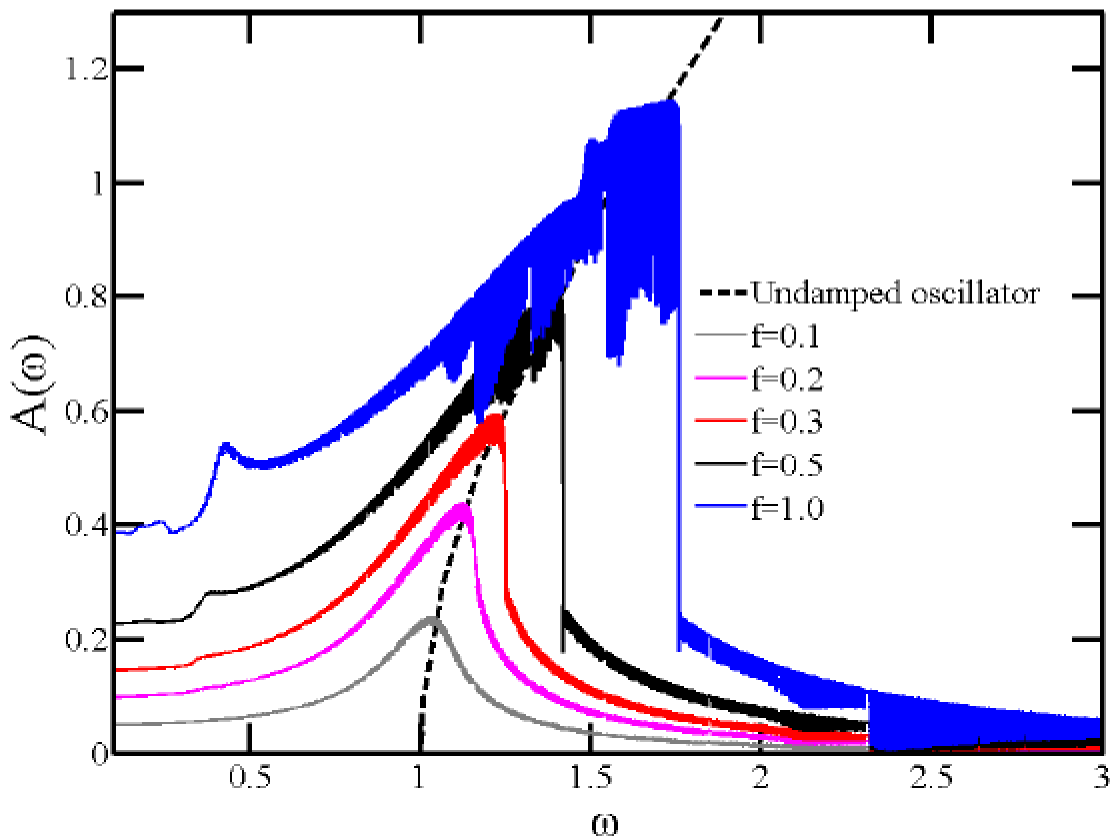
$$A = \frac{f/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega/m)^2}} = \frac{f}{\sqrt{(2\alpha - m\omega^2)^2 + (\gamma\omega)^2}}$$

$$f \sim \sin\phi(t)$$

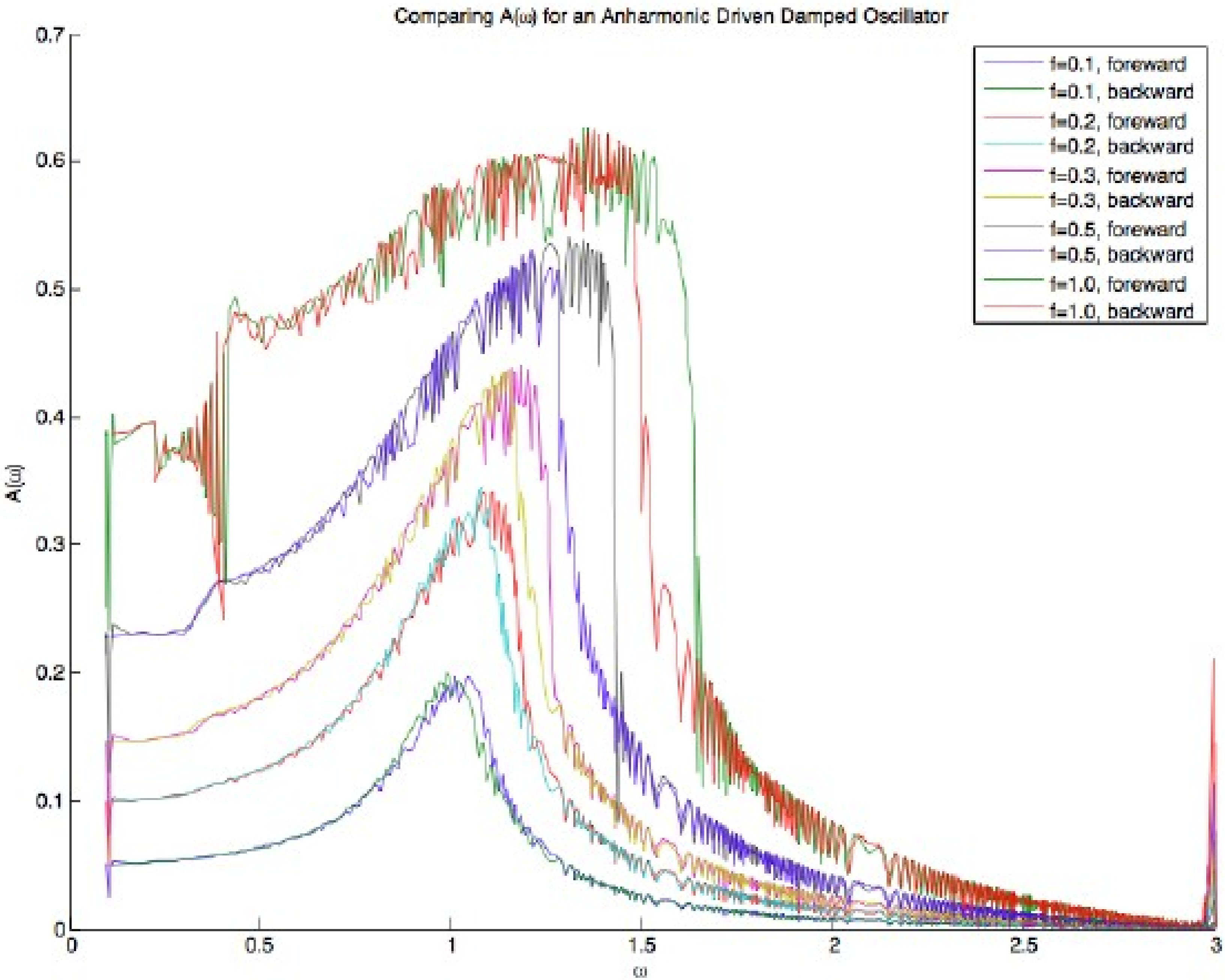
Not $\sin(\omega(t)t)$

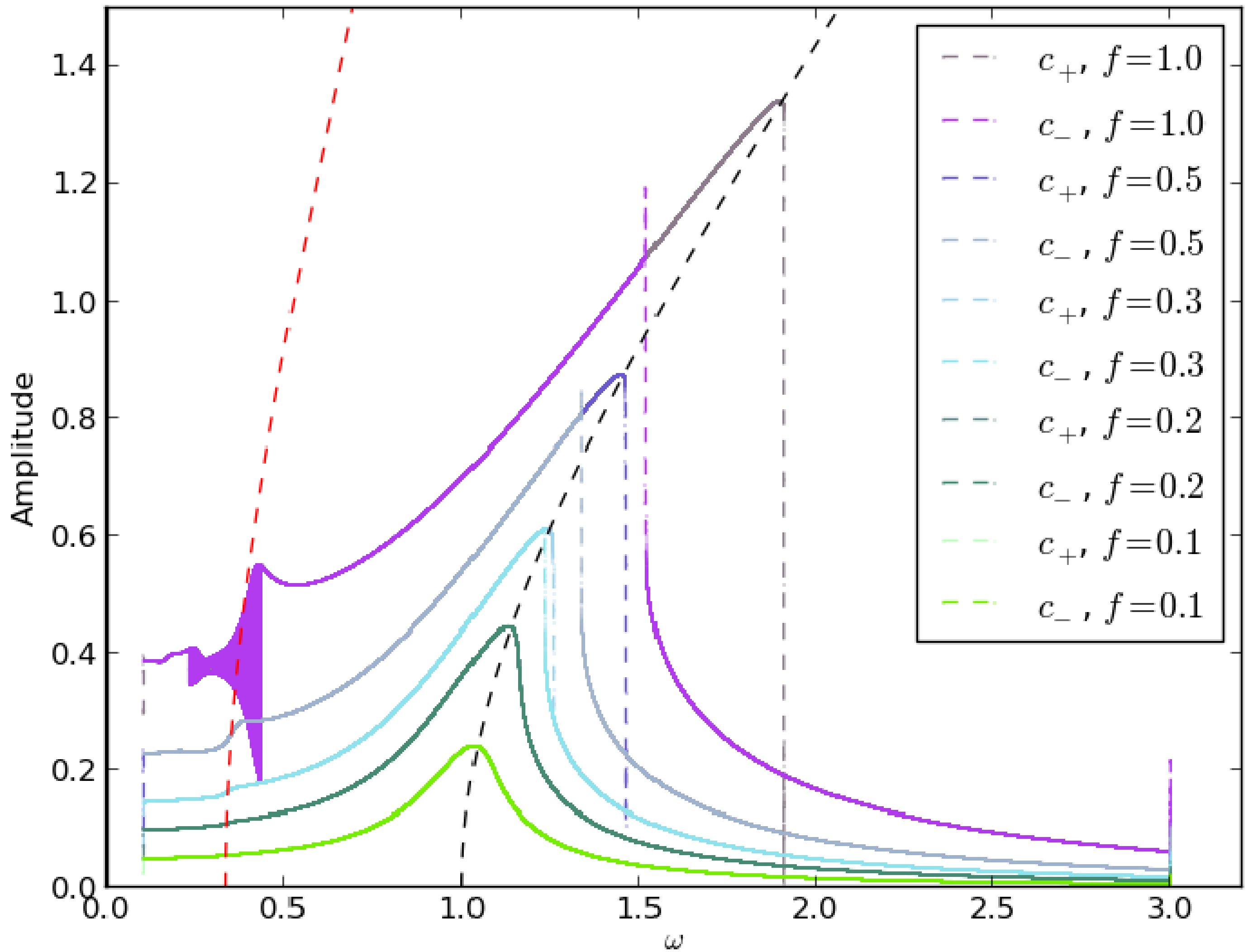


0.1
0.2
0.3
0.5
1.0
A
ω
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0.5
1.5
2.0
2.5
3.0

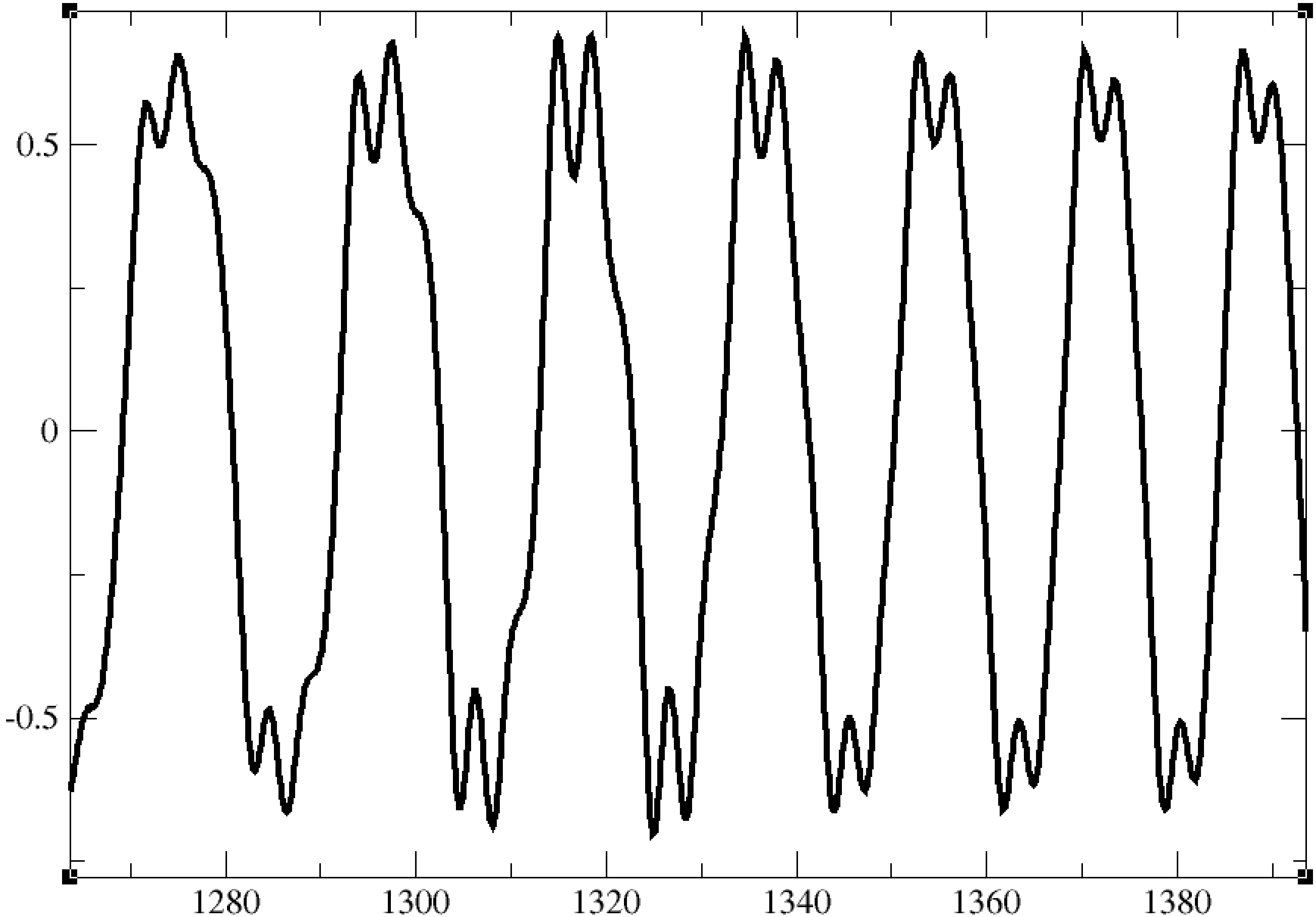


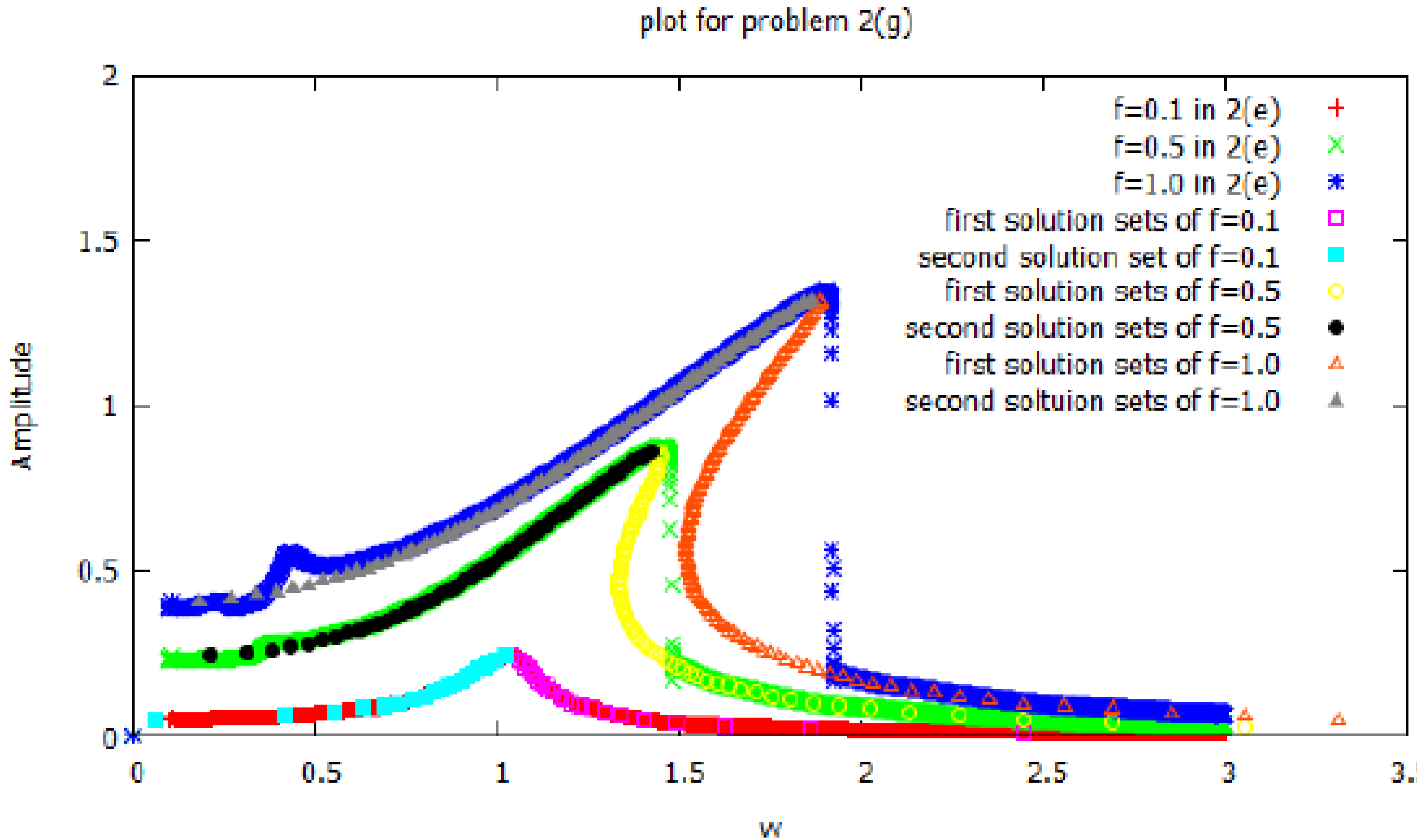
A(ω)
ω
1.2
1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5
3
Undamped oscillator
f=0.1
f=0.2
f=0.3
f=0.5
f=1.0


Comparing A(ω) for an Anharmonic Driven Damped Oscillator
A(ω)
ω
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.5
1
1.5
2
2.5
3
f=0.1, foreward
f=0.1, backward
f=0.2, foreward
f=0.2, backward
f=0.3, foreward
f=0.3, backward
f=0.5, foreward
f=0.5, backward
f=1.0, foreward
f=1.0, backward

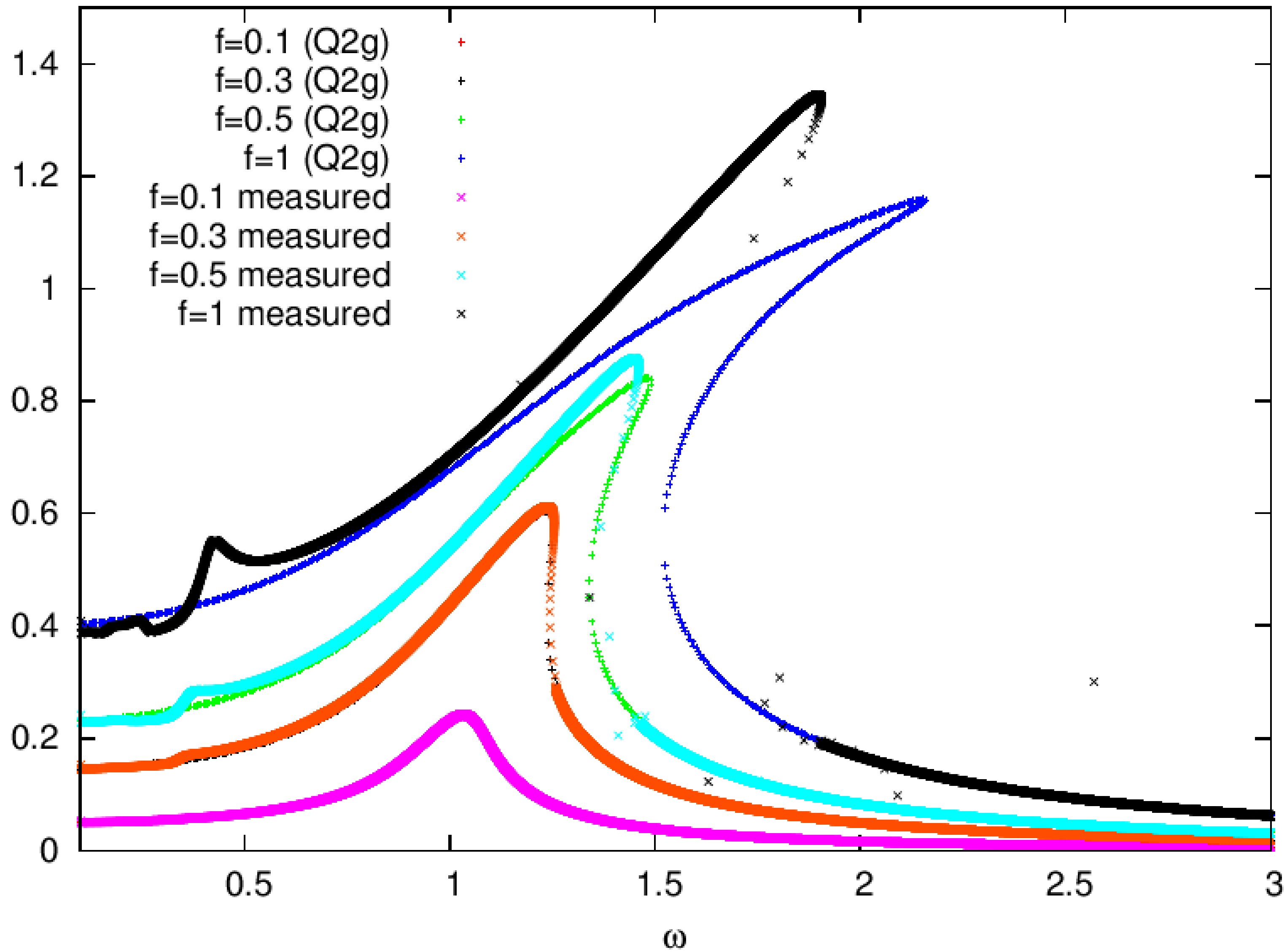


Amplitude
ω
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0.0
0.5
1.0
1.5
2.0
2.5
3.0
c_+, $f=1.0$
c_- , $f=1.0$
c_+, $f=0.5$
c_- , $f=0.5$
c_+, $f=0.3$
c_- , $f=0.3$
c_+, $f=0.2$
c_- , $f=0.2$
c_+, $f=0.1$
c_- , $f=0.1$


0.5
0
-0.5
1280
1300
1320
1340
1360
1380



plot for problem 2(g)
Amplitude
w
0
0.5
1
1.5
2
2.5
3
3.5
f=0.1 in 2(e)
f=0.5 in 2(e)
f=1.0 in 2(e)
first solution sets of f=0.1
second solution set of f=0.1
first solution sets of f=0.5
second solution sets of f=0.5
first solution sets of f=1.0
second soltuion sets of f=1.0



f=0.1 (Q2g)
f=0.3 (Q2g)
f=0.5 (Q2g)
f=1 (Q2g)
f=0.1 measured
f=0.3 measured
f=0.5 measured
f=1 measured
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.5
1
1.5
2
2.5
3
ω