Laboratory 3

Due Friday, October 4, 2013

Just as in the previous lab, consider a point particle of mass m moving in the potential

$$U(x) = \alpha x^2 + \beta x^4, \tag{1}$$

where $\alpha > 0$ and $\beta \geq 0$ are constants.

1. For the first part of the problem, assume that the only force acting on the particle is due to the potential $U(x)$ [Eq. (1)].

(a) (1 pt.) Write down the differential equation describing the motion of the particle.

(b) (2 pts.) Write a program solving this differential equation numerically for given m, α, β and the amplitude A, assuming that the initial position $x(0) = 0$. Choose what to output according to the tasks that you need to perform (see below). Describe your choice in the report. If you find it convenient to write separate programs for each of the tasks, you are free to do so, but make sure you include all the programs with your submission.

(c) (3 pts.) Fix $m = 2$, $\alpha = 1$, $\beta = 1$. Using the code developed in 1(b), calculate the period of the oscillator for $A = 0.01$, 0.1, 0.3, 0.5, 1.0, 3.0, 5.0, 10.0, 30.0, and 100.0. In each of these cases, you will need to achieve the same accuracy as in the previous lab, i.e., $|\delta T/T| \leq 10^{-6}$. Describe your procedure in detail for each of these cases explaining why you think you have achieved the necessary accuracy (you **cannot** refer to the results of the previous lab in your explanation). Now, compare the values to the results from the previous lab (if you do not have the results from the previous lab for some of these values of A, skip those values). Do the results agree?

(d) **(Optional for PHY4140 students)** (2 pts.) As you have seen, the period of the oscillations depends significantly on their amplitude. Is the same true about the **shape** of the $x(t)$ dependence? To find out, plot x/A vs. t/T for $A = 100$ and t/T from 0 to 2 and show **in the same plot** the dependence expected *theoretically* in the limit $A \to 0$. Make your conclusions based on the plot and describe them in the report.

2. In the second part, we consider a *driven damped oscillator*. That is, in addition to the force due to the potential $U(x)$ [Eq. (1)], there is also a damping force, $-\gamma\dot{x}$, and a driving force $F(t) = f\sin(\phi(t))$, where f is a

constant. The instantaneous frequency of the driving force, $\omega(t) \equiv \dot{\phi}(t)$ is allowed to change slowly, as described below.

(a) (3 pts.) Write a program that solves the above problem numerically for given m, α and β assuming $x(0) = 0$, $\dot{x}(0) = 0$. The force $F(t)$ is specified as follows: the initial instantaneous frequency $\omega(0) = \omega_0$ is given; this frequency stays constant for a specified "equilibration time" t_{eq} (that you can round to the nearest integer number of steps of your numerical method) and then changes linearly at a given rate c (that can be positive or negative) until a given frequency ω_f is reached, at which point the computation stops (of course, $\omega_f > \omega_0$ when $c > 0$ and vice versa when $c < 0$). That is,

$$\omega(t) = \begin{cases} \omega_0, & \text{if } t < t_{\text{eq}}, \\ \omega_0 + c(t - t_{\text{eq}}), & \text{if } t_{\text{eq}} < t < t_{\text{eq}} + (\omega_f - \omega_0)/c. \end{cases} \tag{2}$$

With the exception of the initial equilibration period, the program should output the instantaneous frequency $\omega(t)$ and the instantaneous amplitude of the oscillations $A(t)$ roughly once every period of the force $F(t)$ [that is, the time interval between the outputs should be about $2\pi/\omega(t)$]. The instantaneous amplitude is defined as the largest value of $x(t)$ during one period of $F(t)$.

(b) (1 pt.) Consider first the case of a *harmonic* oscillator ($\beta = 0$) and $\omega(t) = \omega = \text{const}$. Derive analytically the dependence of the amplitude of the driven oscillations (assuming that the steady state is reached) on the frequency ω. Your equation should contain f, m, γ, and the frequency of the oscillator $\omega_0 = \sqrt{2\alpha/m}$.

(c) (2 pts.) Numerically, rather than obtaining the dependence in 2(b) point by point (separately for each frequency), one can sweep a range of frequencies in one run, and if the rate of the frequency change c is small, the instantaneous amplitudes at each instantaneous frequency should be the same as for $\omega = \text{const}$. Consider the case $m = 2$, $\alpha = 1$, $\beta = 0$, $\gamma = 0.4$, $f = 1$. Start at $\omega_0 = 0.1$ and stop at $\omega_f = 3.0$. Choose the equilibration time $t_{\text{eq}} = 500$ and the rate of frequency change $c = 10^{-5}$. Since your output will be a table with the (instantaneous) frequency and the corresponding (instantaneous) amplitude, you can compare directly to the equation derived in 2(b). Estimate the error as the maximum discrepancy between the numerical result and the analytical one. The step of your numerical method should be such that its further decrease leads to a change in your results that is small compared to the error you have estimated. However, a step that is too small will make your code inefficient. Choose a reasonable step size and use it for the rest of this lab. Repeat in the opposite direction, starting at $\omega_0 = 3.0$, ending at $\omega_f = 0.1$ and using $c = -10^{-5}$, and compare the results obtained in the two directions.

(d) (**Optional for PHY4140 students**) (2 pts.) Check that the equilibration time t_{eq} is sufficient. Can it actually be reduced without sacrificing the accuracy? By how much? How does the discrepancy between the analytical and numerical results depend on the rate c?

(e) (3 pts.) Now, consider the anharmonic case. Use $m = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.4$. Use the same ω_0, ω_f and c as in the harmonic case. Make sure you go both forward ($c = 10^{-5}$) and backward ($c = -10^{-5}$). Consider $f = 0.1$, 0.2, 0.3, 0.5, and 1. Plot all the results on one plot. On the same plot, also draw as a dashed line the dependence $A(\omega)$ for the free undamped oscillator using the table of $T(A)$ that you have obtained in the previous lab. Describe the results qualitatively. Are they still the same when the frequency increases and when it decreases? Does the amplitude always change smoothly or are there jumps?

(f) **(Optional — 1 bonus point)** Do you see some additional peaks besides the main one on some of the curves? What do you think are they? [Hint: plot $A(3\omega)$ (based on the same data from the previous lab) in the same plot.]

(g) **(Optional — 3 bonus points)** One way to interpret the results you have obtained in 2(e) for the anharmonic oscillator is by using the analytical result of 2(b) for the harmonic one replacing the constant ω_0 in it with the amplitude-dependent $\omega_0(A) = 2\pi/T(A)$, where $T(A)$ is the one obtained in the previous lab. Instead of a closed expression for $A(\omega)$ one then gets an equation that may have multiple solutions. Plot these on the same plot you have obtained in 2(e) (if it now looks too messy, skip some values of f), and describe the results.

(h) (1 pt.) Note that in 2(c) and 2(e), since we used a rather low value of $|c| = 10^{-5}$, we had to solve a differential equation over a long time interval ($\sim 10^4 - 10^5$ cycles). As discussed in class, in some cases this is problematic for some methods (e.g., 4th order Runge-Kutta), and symplectic methods have to be used instead. Is this an issue here? Why or why not?

(i) **(Optional — 2 bonus points)** Study the sensitivity of the results in 2(e) to the value of $|c|$ by varying it. Is the value you have used (10^{-5}) low enough to approximate the limit of an infinitely slow frequency change? You can concentrate on the most interesting parts of the curves for this study.