Laboratory 1

Due Monday, September 16, 2013

Problem 1

Consider a quadratic equation

$$x^2 - bx - 1 = 0, \tag{1}$$

where $b > 0$. As you know, the smaller of the two solutions is given by

$$x_1 = \frac{b - \sqrt{b^2 + 4}}{2}. \tag{2}$$

Another way to express the same solution is

$$x_1' = -\frac{2}{b + \sqrt{b^2 + 4}}. \tag{3}$$

1. (1 pt) Show analytically (i.e., on paper, not on a computer) that Eqs. (2) and (3) are mathematically equivalent.

2. (3 pts) Write a program that for different b calculates the solution of the quadratic equation (1) using **both** Eqs. (2) and (3). Start with $b = 1$, increase b with a 10% increment (i.e., each next b should be 10% larger than the previous one) until you reach $b \sim 10^{15}$. Use double precision floating-point numbers in your program. The output of your program should be a file with the values of b in the first column and the corresponding values of x_1, x_1', and $x_1 - x_1'$ in the second, third and fourth columns (you can add more columns if you find it convenient, e.g., for plotting purposes — if you do, mention this in the `README` file or in your report). Make sure you include the output file when you submit this lab/homework!

3. Using the output of your program, answer the following questions:

(a) (3 pts) Are the values of x_1 the same as x_1' for all b? If not, why are they different, even though the expressions are mathematically equivalent? For what range of b (e.g., small, large, intermediate) is the relative difference

$$r = |(x_1 - x_1')/x_1'| \tag{4}$$

the smallest? The largest? Why? Which one of the two do you think is the more accurate solution of Eq. (1), x_1 or x'_1? Why do you think so?

(b) (1 pt) Is there a range of b where x_1 (but not x'_1) is zero? Why do you think this is the case?

(c) (2 pts) Plot r vs. b on a log-log plot. Describe the plot qualitatively. Do you find the behaviour qualitatively different in different ranges of b?

(d) (2 pts) Is there a range of b where the dependence is approximately linear on the log-log scale? Note that if $\log r = A \log b + \log C$, where A and C are constants, then $r = Cb^A$, and therefore the dependence $r(b)$ is a power law with the slope A on the log-log plot equal to the exponent. What is this slope? (You can either do a fit or draw a straight line by eye.)

(e) **(Optional for PHY 4140 students)** (3 pts) Explain the value of the exponent A you have obtained in (c). Can you also explain the prefactor C? (Note that in order to fully understand this you **may** have to take into account the fact that depending on the processor and the compiler, **some** mathematical calculations, like the square root operation, **may be** done in 80-bit extended precision with $\epsilon \approx 5 \times 10^{-20}$, instead of double precision with $\epsilon \approx 10^{-16}$.)

Problem 2

The area of a circle of unit radius can be expressed as

$$4 \int_0^1 \sqrt{1 - x^2} dx. \tag{5}$$

1. (4 pts) Write a program calculating the integral (5) using the trapezoidal rule for different numbers of grid points (with a constant grid spacing) and estimating the error using the known result for the area of a circle. Use **single precision** floating point numbers, if you can. Vary the number of grid points N between two (just the ends of the interval) and 100. The output of your program should be a file with three columns: the number of points N (or, alternatively, the grid spacing h), the corresponding value of the integral, and the error (the difference between the calculated value and the exact value).

2. (2 pts)

(a) Plot the dependence of the absolute value of the error on h on a log-log plot.

(b) Fit the dependence with a power law (either using fitting software or by eye). Is the exponent the same as predicted in class for the trapezoidal rule? If not, what is the reason?

3. **(Optional for PHY4140 students)** (2 pts) Explain the value of the exponent you've obtained.

4. **(Optional - 2 bonus points)** If you've used single precision in your code, extend your plot to $N = 10^7$. What do you observe for small h? Have you seen something similar in class? More bonus points if you can explain the exponent observed for small h.