Mean-field conductivity in a certain class of networks

M. V. Chubynsky[1,2,*] and M. F. Thorpe[2,†]
[1]*Département de Physique, Université de Montréal, Case Postale 6128, Succursale Centre-ville, Montréal, Québec, Canada H3C 3J7*
[2]*Department of Physics and Astronomy, Arizona State University, P. O. Box 871504, Tempe, Arizona, 85287-1504, USA*
(Received 15 November 2004; revised manuscript received 7 March 2005; published 11 May 2005)

We consider resistor networks, which are lattices with bonds represented by conductors and some of the bonds removed. It is known that effective medium theories predict that the effective conductivity of such networks is a linear function of the number of bonds present above the percolation threshold, but exact results for completely random networks deviate from linearity. We show that if instead we take a randomly chosen tree spanning the lattice and then start adding bonds to it at random, the conductivity changes linearly with the number of added bonds and coincides with the effective medium result for a given bond concentration. We also make comparisons with some related models.

DOI: 10.1103/PhysRevE.71.056105 PACS number(s): 64.60.Fr, 64.60.Ak, 02.10.Ox

I. INTRODUCTION

Percolation has been widely studied as a relatively simple model of phase transitions and critical phenomena having many of the features of these phenomena in more complex systems [1]. The simplest percolation model is random bond-diluted percolation, in which case one starts with a regular lattice and then removes bonds, choosing them completely at random. The diluted networks are characterized by the fraction of bonds present (*bond concentration*) p. In the limit of an infinite lattice (*thermodynamic limit*), for p above the percolation threshold p_c, an infinite percolating cluster (a set of mutually connected bonds) emerges, while below p_c only finite clusters exist.

A diluted network can be made into a random resistor network, if the bonds present are replaced with resistors all having the same resistance. Then one can apply a potential difference across the network and find the current, thus measuring the electrical conductance of the network. One can also introduce the effective conductivity as the conductivity of a uniform material of the same size and shape as the network having the same conductance. Below the percolation threshold, the conductivity is zero, as the opposite sides are not connected; as p grows and the threshold is crossed, the conductivity starts growing from zero. A mean-field (or effective medium) theory (MFT) exists for conductivity [2], which predicts that this growth occurs linearly as a function of $p-p_c$, with $p_c=2/z$, where z is the coordination number of the undiluted lattice. This works well far away from p_c; in fact, the predicted slope is always correct at $p=1$ [3].

Close to p_c, however, the conductivity dependence is generally nonlinear, with the critical exponent depending on the dimensionality [e.g., $\simeq 1.30$ in two dimensions (2D) [1]], but independent of the lattice type (e.g., square, triangular, etc., in 2D). There are very few cases where the MFT for some quantity is an exact solution. Above the upper critical dimension for a given universality class, the critical exponents are the same as in the MFT [4] (although the details of the dependence may be different). Also, mean-field or mean-field-like behavior is seen in systems with long-range correlations, such as spin systems, in which all pairs of spins interact equally, irrespective of how far apart they are [5], or systems on rather "pathological" Bethe lattices [6]. In these examples, switching to mean-field behavior requires drastic changes, such as going to higher dimensionality or introducing rather unphysical long-range interactions. In the present paper we show that even fairly weak geometric correlations can lead to mean-field behavior for the conductivity in random media.

II. THE MODEL

Consider the following model of a correlated network. First introduce the concept of a *spanning tree*, which is a network that has no loops (i.e., there is never more than one path connecting any two sites) and in which all sites are connected. In other words, in a spanning tree exactly one path connects any pair of sites. Suppose we have a set of all possible spanning trees on a given lattice and choose one of them at random. Such a tree is called a uniform spanning tree (UST) reflecting the fact that it is chosen with a probability uniform among all trees. It is clear that such a tree is a network with long-range correlations, as whether or not a certain bond closes a loop can depend on the presence of other bonds infinitely far away from it. The conductivities of trees are zero in the thermodynamic limit, as with no loops there are very few connections between opposite boundaries. In fact, conductivity can be zero even for finite samples, in the case of *periodic boundary conditions* (PBCs). PBCs are introduced by considering opposite sites at opposite boundaries of the lattice as identical (thus effectively wrapping the lattice into a "torus" of appropriate dimensionality) when building a spanning tree, but then requiring that the potential differs by a specified constant at opposite sites in one direction and is identical in other directions (for a 2D lattice, imagine an ordinary torus in 3D with a variable magnetic flux through it with a constant rate of change). Then for conductance one needs the presence of loops around the

*Electronic address: mykyta.chubynsky@umontreal.ca
†Electronic address: mft@asu.edu

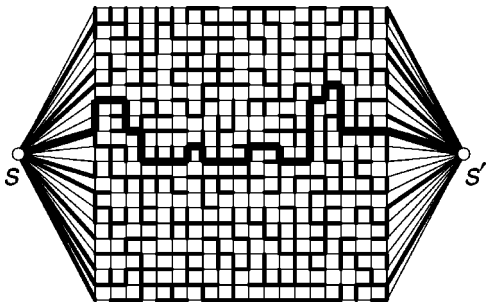

FIG. 1. A tree built on the 20×20 square lattice with a source (s) and a sink (s') added. The missing bonds are shown with thin lines, the present bonds (forming the tree) with thicker lines, including the only path from s to s' shown with the thickest lines. The missing bonds are then added at random, as described in the text.

"torus" in the direction of potential change; but there are no loops whatsoever in a tree, and thus the conductance is indeed zero even in finite samples. This geometry is convenient for numerical simulations, as described below. A different setup, the source-sink geometry, is more convenient for theoretical analysis that we present later in this paper. In this case, we have open boundary conditions, but put two additional sites, a *source* and a *sink*, at the opposite boundaries (Fig. 1), allow connections between these sites and the adjacent boundary, and measure the conductance between the source and the sink; in this case there is just one connection and the conductivity is again zero in the thermodynamic limit.

Our model consists in starting from a UST and adding bonds, choosing their places at random among those where they are missing. The bonds present represent unit resistances. As bonds are added, the conductivity starts to grow and it is the dependence of the conductivity on the number of placed bonds (or on the bond concentration p) that we are going to study. We will call this model the addition to the UST (AUST) model in what follows.

Consider a tree spanning a lattice of N sites. When a tree is being built, every new bond must connect two clusters, joining them together into one cluster (new bonds cannot join sites already in the same cluster, or else they would close a loop); thus each bond decreases the number of clusters by 1, and given that there are N clusters in an empty network with no bonds and just one cluster in the spanning tree, the number of bonds is $N-1$. This corresponds to the bond concentration $p=2/z$ (as a reminder, z is the coordination number of the full lattice). If we make an assumption (that will actually follow from the proof below) that the conductivity starts growing from zero immediately as the bonds start to be added to the tree, the value $p=2/z$ should be regarded as the analog of the percolation threshold. We note that it coincides with the mean-field value [2].

III. NUMERICAL RESULTS

First, the study of conductivity in our AUST model can be done numerically. We did this by solving the system of Kirchhoff equations directly by the conjugate gradient method [7]. We preferred this to somewhat faster special methods, such as those using the star-triangle transformation [8], as we could treat periodic boundary conditions. As we mentioned, in this case the conductance of a tree is exactly zero even for a finite sample, so the finite size effects are expected to be less severe; of course, we expect that in the thermodynamic limit the results do not depend on the boundary conditions and in particular, are the same for PBCs and in the source-sink geometry. The starting uniform trees are built using an algorithm due to Broder [9]. In the Broder algorithm, a random walk on the network is started at an arbitrary site and continued until all sites are visited; every time a new site is reached, the last bond along which the walk reached that site is recorded; the set of all such bonds (there are $N-1$ of them, since one bond is recorded for each site but the starting one) forms a tree and it can be proved [9] that every possible tree will occur with equal probability. In Fig. 2 we compare the results for the usual randomly diluted lattices with those for our AUST model. In Fig. 2(a), we present the results for the square lattice. In this case, the comparison is facilitated by the percolating thresholds being the same in the two models; indeed, for the randomly diluted

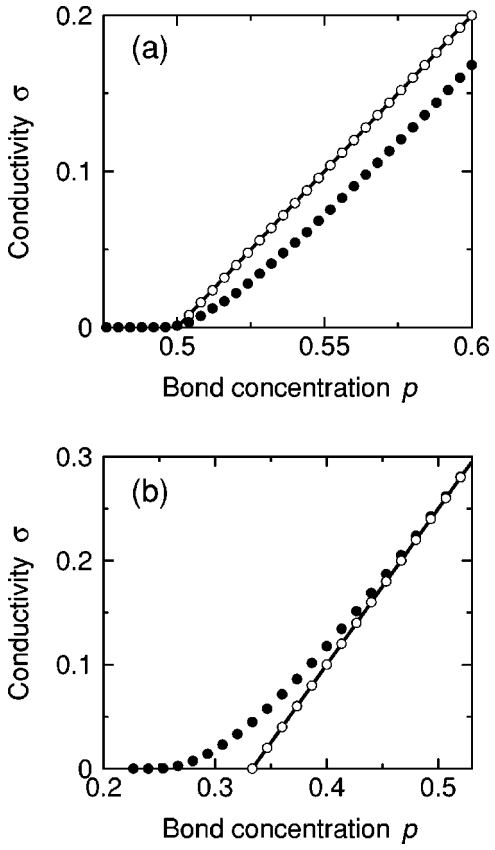

FIG. 2. Conductivity σ as a function of bond concentration p for random bond-diluted networks (filled circles) and our correlated AUST model (open circles) on the square lattice (a) averaged over 25 realizations on the 500×500 lattice and on the simple cubic lattice (b) averaged over 25 realizations on the $50\times50\times50$ lattice. The solid lines are the respective effective medium linear results. In all cases, $\sigma=1$ at $p=1$. For AUST results, a different starting spanning tree generated at random was used for each realization.

square lattice, the threshold is known to occur at the mean-field value ($p_c = 0.5$ in this case); as we mentioned, this is the case for the AUST model, regardless of the lattice. In Fig. 2(b), we do the same comparison for the simple cubic lattice. In this case the thresholds do not coincide: the mean-field (and AUST) value is $1/3$, the randomly diluted network value is 0.2488 [1].

The striking feature of the results in Fig. 2 is that while in the random dilution case the deviation from linearity is seen very well close to the percolation threshold, the dependence for the AUST model seems to be linear. Note that this is the case in both 2D [Fig. 2(a)] and 3D [Fig. 2(b)]. Of course, the question is if this dependence is indeed exactly linear or just very close to linear, and we are going to show now that this linearity is *exact*.

IV. PROOF OF LINEARITY OF CONDUCTIVITY

We will use a result due to Kirchhoff [10,11] that expresses the resistance between two points of a resistor network in terms of sums over trees built on the network. In our particular case, when all resistances are equal to unity, this result reduces to a tree-counting procedure. First, find the total number N_T of distinct spanning trees that can be built on the network [Fig. 3(b)]. Second, find the total number $N_{T'}$ of possible distinct graphs consisting of two trees, such that the source is in one tree and the sink in the other [Fig. 3(c)]. Following Bollobás [12], we will refer to such two-tree graphs as *thickets*. Then the conductance S is the ratio of these two numbers:

$$S = \frac{N_T}{N_{T'}}. \tag{1}$$

This result is based on the following fact. Suppose there is a spanning tree on the network and one drives a unit current between the source s and the sink s' (the current will, of course, be confined to the "backbone" of the tree, which is just a single path, as illustrated in Fig. 1). Now, if one repeats this for all possible spanning trees and forms the superposition of the currents, then the resulting current distribution will satisfy all Kirchhoff laws for the original network with no other sources and sinks but s and s'. That the sum of currents at each node is zero is obvious, since this is so for each separate tree and thus also holds for the superposition; checking that the voltage drop around any loop is zero is a bit trickier, but can be done by expressing the current in each loop in terms of certain thickets and then looking at the contribution of each thicket (for details, see Ref. [12]). Once this fact is established, one can consider a network consisting of the original one plus an additional link of unit resistance between s and s'; the conductance of the original network is then the ratio of the current through the original network and the current through the added link (the latter being equal to the voltage between s and s').

For our proof, it is convenient to visualize the following diagram (Fig. 4). First, imagine we have a set of all spanning trees that can be built on the full lattice. This set is denoted schematically as the left column of dots in Fig. 4. Now consider a set of all networks that can be obtained from these trees by adding exactly B bonds (the middle column of dots in Fig. 4). From each tree we can obtain

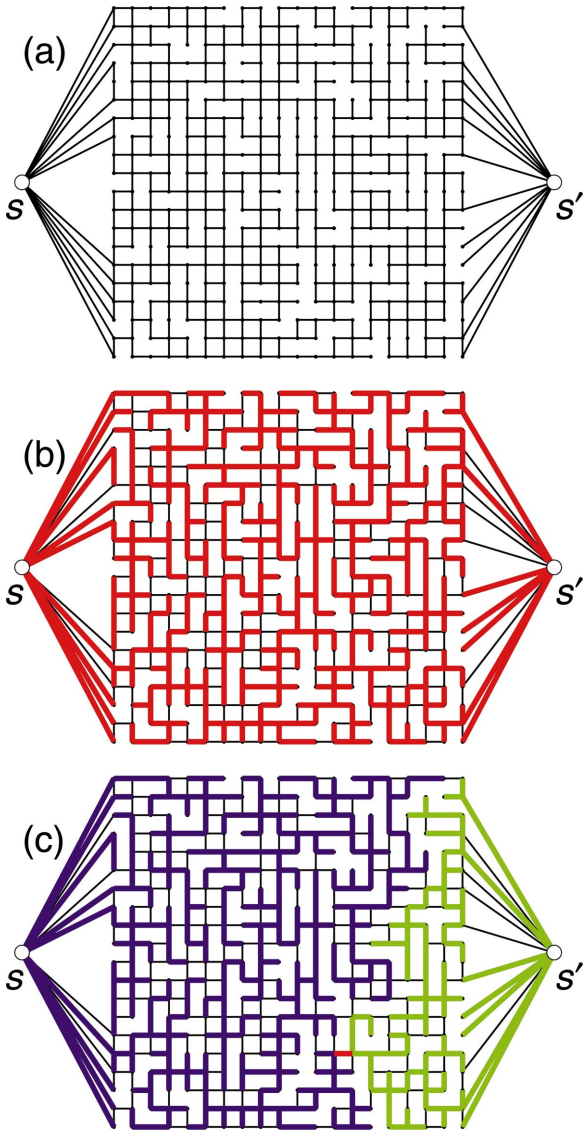

FIG. 3. (Color) An illustration of the relation between networks, trees, and thickets. (a) shows a network spanning all sites of the square lattice (such as would be obtained by adding bonds to a spanning tree). (b) illustrates a tree (red) built on the network in (a), or, conversely, the network in (a) can be obtained by adding bonds to the tree in (b). Finally, (c) shows a thicket consisting of two trees (blue and green) on the network in (a). Note that the particular thicket in (c) can be obtained by removing a single bond [red in (c)] from the backbone of the tree in (b).

$$N_n = \binom{B_0}{B} \tag{2}$$

different networks, where B_0 is the total number of bonds missing in the tree compared to the full lattice (note that B_0 is the same for all trees, as the number of bonds in any tree is $N-1$). This establishes connections between the set of trees and the set of networks, so that every tree is connected to N_n

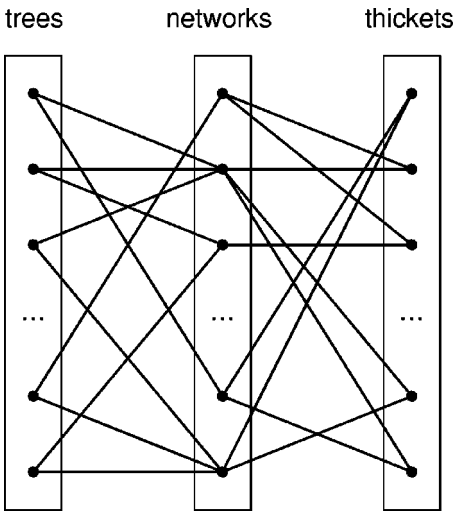

FIG. 4. A schematic diagram showing relations between trees and networks and between thickets and networks. The left column of dots denotes the set of all possible trees on the full lattice; the right column is the set of all thickets; the middle column is the set of all spanning networks with a certain number of bonds. The connections in the left part (between trees and networks) show what networks can be built by adding bonds to a tree, or, conversely, what trees are subgraphs of a given network. The connections in the right part show similar relations between thickets and networks. The ratio of the numbers of connections in the left part and in the right part is proportional to the conductivity of a network with a certain number of bonds in the thermodynamic limit, as discussed in the text.

networks that can be obtained from it. These connections are shown schematically in Fig. 4 as the links between the dots in the left column and the dots in the middle column. The total number of connections is

$$C = N_n T_f, \tag{3}$$

where T_f is the total number of possible spanning trees on the full lattice. Conversely, these same connections specify which trees can be built on each network (i.e., are subgraphs of the network).

Similarly, we can consider the set of all possible thickets on the full lattice (denoted by the right column of dots). Every thicket has $N-2$ bonds, so there are B_0+1 empty bonds. If we add a set of $B+1$ bonds to a thicket, we will almost certainly obtain one of the networks under consideration (there is a possibility that none of these bonds connect the two trees of the thicket together, but this is negligibly rare in the thermodynamic limit). Then the number of networks that can be obtained from each thicket by bond insertion is

$$N_n' = \binom{B_0 + 1}{B + 1} \tag{4}$$

and the total number of connections is

$$C' = N_n' T_f', \tag{5}$$

where T_f' is the total number of possible thickets on the full lattice. These connections are also shown in the diagram. Again, they also specify which thickets can be built on each network.

A plausible assumption that we make is that conductance is self-averaging, i.e., in the thermodynamic limit the conductance is the same for all but a negligible fraction of realizations. We verified this assumption numerically, by checking that the variance in the conductance for different realizations at the same bond concentration decreases as the network size increases. Then for almost every network the ratio of the number of trees connected to it to the number of thickets connected to it is the same [as this ratio equals the conductivity, according to Eq. (1)]. The anomalous networks, for which this is not the case, have a negligible probability of occurring. The probability to obtain a particular network is proportional to the number of connections between this network and various trees; thus anomalous networks have a negligible amount of connections with the trees; this is also true for their connections with the thickets, as the ratio of the number of connections with the trees to that with the thickets for every network is its conductance, and the conductivity is expected to be $O(1)$ for all but an exponentially small fraction of networks. Then all connections of the anomalous networks can be neglected in the total count of connections and the ratio of the total number of connections between the left and the middle columns to that between the right and the middle columns is again the conductance. On the other hand, this ratio is

$$S = C/C' = \frac{N_n T_f}{N_n' T_f'} = \frac{T_f (B+1)}{T_f' (B_0+1)} \propto B \quad \text{for } B \gg 1, \tag{6}$$

as B_0, T_f, and T_f' do not depend on B, and the proof is complete.

Some comments are in order. First, we have not made any assumptions about the underlying lattice, so that the result is independent of the lattice type and dimensionality, although there may be problems with the assumptions that we made in pathological cases, when connections between sites infinitely far apart are possible. Second, we had in mind a situation with the source and the sink at the opposite sides of the lattice, but have not used this fact anywhere. We did make an assumption that conductance is realization independent and this is only true when the source and the sink are infinitely far apart.

V. RELATED MODELS

Besides UST, it is possible to produce various *biased* distributions of spanning trees. One much-studied example is the minimal spanning tree (MST) [13], which is, given a lattice with weights assigned to bonds, the tree with minimal total weight. Another example of a spanning tree is the shortest-path tree (SPT) [13], which, again given the weights of bonds, is the set of paths with minimum weight between a particular site and all other sites. Note that our proof does not apply to these biased cases, as atypical networks with

anomalous conductivity may be strongly favored over typical ones. Indeed, the trees themselves are atypical: the geometries of the MST and SPT are different from those of the typical UST; this can be characterized, e.g., by the fractal dimension of paths on the trees, which in 2D is $5/4$ for UST [14], $\simeq 1.22$ for MST [15], and 1 for SPT for moderate disorder in weights [16]. Thus branches of a SPT are much more straight than those of a UST. Interestingly, though, it turns out that the conductivity dependence is still surprisingly close to linear even for the SPT, despite their being geometrically very different from the UST. For example, for the square lattice, at $p=0.55$ the conductivity of networks obtained from the SPT is only about 3% above that of networks obtained from the UST. We also note that for the MST, the deviations from linearity have never been reliably detected either in 2D or in 3D. Thus it may be possible that the linearity is exact for the MST, just as for the UST. The reasons for this linearity or near-linearity in the MST and SPT are not clear at the moment.

At the same time it is easy to produce spanning networks, for which their conductivity dependence is strongly nonlinear. Trivial examples can be obtained by adding bonds to anisotropic trees, in which most branches are directed, say, perpendicular to the applied potential difference. More interesting is the following *isotropic* case (called the restricted dilution model in what follows). Start from the full lattice and start picking bonds at random, but removing them only if this removal does not separate a piece of the network from the rest. In this way, we obtain spanning networks, with no finite clusters, just as in our AUST model. Obviously, this dilution procedure can be continued until a spanning tree is obtained. Despite certain similarity with the AUST model, the conductivity dependence now is not mean-field-like. Indeed, it can be shown that the conductivity becomes zero *before* reaching the spanning tree limit, i.e., at a higher bond concentration (Fig. 5). Thus, an interesting example of an *intermediate phase* is formed: in a certain range of bond concentrations, the infinite cluster exists, but its structure is such that the conductance is very low and vanishes in the thermodynamic limit.

To see this, suppose we create a random list of bonds intended for removal. In the case of random bond dilution, all of these bonds are removed in the order given by the list. In the restricted dilution case, some of these bonds will be rejected and not removed. Note that rejected bonds do not belong to loops and so their removal would not change the configuration of loops and thus would not change conductivity (at least not in the case of PBCs). Then, there is a correspondence between sequences of networks obtained by random dilution and those obtained by restricted dilution using the same list of bonds whose removal is attempted. If the random dilution procedure proceeds below the percolation threshold, the conductivity of obtained networks becomes zero, so the conductivity of the corresponding networks obtained by restricted dilution is also zero, even though by construction these networks still span all sites.

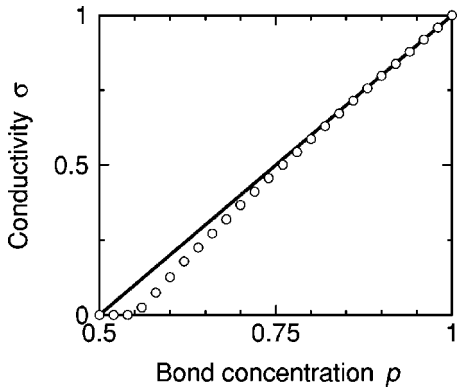

FIG. 5. Conductivity as a function of bond concentration for the restricted dilution model described in the text (circles). These results are averaged over 25 realizations on the 100×100 square lattice. The solid line is the mean-field linear result.

VI. OUTLOOK

While this paper provides a formal proof of linearity of conductivity in the AUST model, an obvious question is if there is a more straightforward physical explanation. Of course, the network is more uniform in a sense compared to a randomly diluted one, as the percolating cluster spans all sites and there are no "holes" in it in the form of isolated finite clusters. Yet, as the example of the restricted dilution model shows, this by itself is not enough to get even close enough to linearity. There seems to be nothing in the model itself that would suggest that it should exhibit mean-field behavior, so the question remains open at present.

As a final note, a model analogous to the AUST, but using MSTs as initial trees to which bonds are added, has arisen as the connectivity analog of our model [17] of self-organization in *rigidity percolation*, as explained in Ref. [18]. In that case, the analog of conductivity is the elastic moduli of the network of elastic springs, and it is worth noting that the deviations from linearity in the critical region are significant [19], whereas the effective medium theory [20] again predicts linearity. Thus mean-field behavior is unique to ordinary (connectivity) percolation.

ACKNOWLEDGMENTS

We would like to acknowledge support from the National Science Foundation under Grants No. DMR-0078361 and No. DMR-0427933 and from the Center for Fundamental Materials Research at Michigan State University. We are also grateful to P. M. Duxbury for useful discussions.

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).
[2] S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).
[3] M. F. Thorpe and W. Tang, J. Phys. C **20**, 3925 (1987).
[4] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1992).
[5] M. Kac, in *Statistical Physics, Phase Transitions, and Superfluidity*, edited by M. Chrétien, E. P. Gross, and S. Deser (Gordon and Breach, New York, 1968), Vol. 1, p. 241; H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York, 1987).
[6] M. F. Thorpe, in *Excitations in Disordered Systems*, NATO Advanced Study Institute, Series B: Physics (Plenum, New York, 1982), Vol. 78, p. 85.
[7] W. H. Press, S. A. Teukolsky, V. T. Vetterling, and B. P. Flannery, *Numerical Recipes in FORTRAN* (Cambridge University Press, Cambridge, U.K., 1992).
[8] D. J. Frank and C. J. Lobb, Phys. Rev. B **37**, 302 (1988).
[9] A. Z. Broder, in *Proceedings of the 30th Annual Symposium on Foundations of Computer Science* (IEEE, Washington, DC, 1989), p. 442.
[10] G. Kirchhoff, Ann. Phys. Chem. **72**, 497 (1847).
[11] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[12] B. Bollobás, *Graph Theory* (Springer-Verlag, New York, 1979), Sec. II, §1, Theorem 1.
[13] T. H. Cormen, C. E. Leiserson, and R. L. Rivest, *Introduction to Algorithms* (MIT Press, Cambridge, MA, 1990).
[14] S. N. Majumdar, Phys. Rev. Lett. **68**, 2329 (1992).
[15] A. A. Middleton, Phys. Rev. B **61**, 14 787 (2000).
[16] N. Schwartz, A. L. Nazaryev, and S. Havlin, Phys. Rev. E **58**, 7642 (1998).
[17] M. F. Thorpe, D. J. Jacobs, M. V. Chubynsky, and J. C. Phillips, J. Non-Cryst. Solids **266–269**, 859 (2000).
[18] M. F. Thorpe and M. V. Chubynsky, in *Properties and Applications of Amorphous Materials*, edited by M. F. Thorpe and L. Tichý (Kluwer Academic, Dordrecht, 2001), p. 61; in *Phase Transitions and Self-Organization in Electronic and Molecular Networks*, edited by J. C. Phillips and M. F. Thorpe (Kluwer Academic/Plenum Publishers, New York, 2001), p. 43.
[19] M. V. Chubynsky, Ph.D thesis, Michigan State University, 2003.
[20] S. Feng, M. F. Thorpe, and E. Garboczi, Phys. Rev. B **31**, 276 (1985).